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RESONANCE TUNNELLING OF WAVES IN A STRATIFIED COLD PLASMA

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CONTENTS

The theory of the tunnelling of waves through a barrier in which the square of the effective refractive index is zero at one boundary and infinite at or near the other is studied. An infinity of the refractive index is called a resonance and so we speak of resonance tunnelling. The sum of the powers in the reflected and transmitted waves is less than the power in the incident wave even in a loss free system where there is no mechanism for the absorption of energy. A formal proof is given that there must be such a disappearance of energy, associated with the solution of the governing equations that is singular at the resonance. The problem of what has happened to the lost energy is discussed. Some previous treatments dealt only with normally incident waves, but this is a degenerate case. The theory is extended to include oblique incidence and some new features are revealed. Some specific examples are worked out as illustrations.

1. Introduction

The propagation of waves normally incident on a stratified medium is conveniently described by the refractive index μ. For obliquely incident waves μ is replaced by a quantity $q_1 - q_2$ explained later, but for the present μ will be used as though it applies for both oblique and normal incidence. In a loss free system where μ^2 is positive the waves are propagated with a real phase velocity, and where μ^2 is negative they are evanescent. A region where μ^2 is negative is sometimes called a 'barrier' or forbidden region, but it is well known that some wave energy can tunnel through. In the most familiar kind of barrier the boundaries are where μ^2 is zero. But it is also possible for one or both boundaries to be where μ^2 is infinite (figure 1*c*). In the present paper we study the case where one boundary has $\mu^2 = 0$ and the other is at or near where $\mu^2 = \infty$. The situation where $\mu \to \infty$ is sometimes called a resonance, especially in plasma physics, and so we speak of resonance tunnelling. The case where $\mu^2 = \infty$ at both boundaries of a barrier can occur and some aspects of it have been studied (Tang, Wong & Caron 1975) but it is not considered further here.

When a wave is incident on a loss free stratified system containing a resonance, it is found that the sum of the energy fluxes in the transmitted and reflected waves is less than the energy flux in the incident wave, even though there is no mechanism for the absorption of energy. Thus there is an apparent loss of energy and it is an intriguing physical problem to explain what has happened to it. Interesting discussions have been given by Stix (1962) and by Ngan & Swanson (1977). A similar effect has been found in the theory of the oscillations of the Earth's magnetosphere, giving magnetic micropulsations. There is a disappearance of energy near the resonant lines of force, and Southwood (1974) and Chen & Hasegawa (1974) have pointed out that the governing differential equations are similar to those for the radio propagation problem.

This general problem was discussed in a previous paper (Budden 1954). The theory was repeated with amendments in a text book (Budden 1961), and since frequent references to this will be needed, it will here be referred to as *R.w.i.*, the initials of its title. This early treatment considered only the degenerate case of normally incident waves, and could thus be used only for an incident wave of infinite lateral extent. Any wave of bounded lateral extent, coming from a source of limited size, can be expressed as an angular spectrum of plane waves (Clemmow 1966) and in a full treatment it is necessary to consider the obliquely incident component plane waves.

The purpose of the present paper, therefore, is twofold. First it examines the basic theory for a loss free medium so as to throw further light on the physical mechanism by which energy is apparently lost. Second it deals with obliquely incident waves and shows how this more general case is related to the special case of normal incidence.

As in the earlier treatment in *R.w.i.*, this paper is concerned almost entirely with cold plasma. The notation and the type of medium used are explained in § 2. Section 3 sets out the governing differential equations for the electromagnetic field components. It introduces the idea of the adjoint fields and the adjoint differential equation. These were examined recently by Suchy & Altman (1975) and have proved to be a most valuable tool in the present study. Series solutions are found in § 4 and it is shown that, for obliquely incident waves there is one solution that is

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singular at the resonance. For normal incidence this solution ceases to be singular and is no longer an independent solution. The construction of a solution which is uniformly independent, and singular at the resonance, whether or not the incidence is normal, is presented in § 5. Section 6 gives the formal proof that this solution must lead to a disappearance of some energy at the resonance, even when there is no mechanism for absorbing energy. Section 7 uses a simple example from *R.w.i.* (§ 21.14) to illustrate this. Up to this point the study has been mainly concerned with the resonance, whether or not there is a wave barrier. The concept of a barrier is used in the context of characteristic waves or modes introduced in § 8. The problem is now conveniently tackled by using coupled wave equations, § 9. A model for use with these equations is described in § 10. Section 11 examines how the properties of a single coupling point are modified when there is a resonance near it. The full problem of a barrier with a resonance beyond one of its boundaries is studied in § 12. Finally in § 13 there is a discussion of what happens to the energy that is apparently lost near a resonance.

2. Notation and description of the model

A Cartesian coordinate system x, y, z is used with the z axis perpendicular to the cold stratified plasma. Instead of z it is convenient to use the dimensionless variable

$$s = kz, \quad k = \omega/c, \tag{1}$$

where ω is the angular frequency and c is the speed of light. There is a superimposed constant magnetic field whose direction cosines are $-l$, $-m$, $-n$. The properties of the plasma are given by the 'principal axis' values ϵ_1, ϵ_2, ϵ_3 (the same as L, R, P used by Stix 1962) obtained when the dielectric constant tensor $\boldsymbol{\varepsilon}$ is diagonalized by the use of complex principal axes (Westfold 1949). When the plasma contains several species of ion, ϵ_1, ϵ_2, ϵ_3 are each expressed as a sum over species, given for example by Stix (1962), Budden & Smith (1974). They are functions of s only. The range of s where they vary appreciably is assumed to be limited so that they each tend to a constant value when s tends to a large positive or negative real value. When, as here, losses by collisions or other damping processes are ignored, ϵ_1, ϵ_2, ϵ_3 are all real and bounded when s is real. They are assumed to be continuous analytic functions of s, and 'slowly varying' in a sense to be explained later, § 9. Let

$$G = \tfrac{1}{2}(\epsilon_1+\epsilon_2)-\epsilon_3, \quad D = \tfrac{1}{2}(\epsilon_1-\epsilon_2). \tag{2}$$

D is the same as used by Stix (1962). Then in the Cartesian system it can be shown that the dielectric constant tensor is

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \tfrac{1}{2}(\epsilon_1+\epsilon_2)-l^2G & -lmG-\mathrm{i}nD & -lnG+\mathrm{i}mD \\ -lmG+\mathrm{i}nD & \tfrac{1}{2}(\epsilon_1+\epsilon_2)-m^2G & -mnG-\mathrm{i}lD \\ -lnG-\mathrm{i}mD & -mnG+\mathrm{i}lD & \tfrac{1}{2}(\epsilon_1+\epsilon_2)-n^2G \end{bmatrix}. \tag{3}$$

If the direction of the superimposed magnetic field is reversed, l, m, n all change sign and $\boldsymbol{\varepsilon}$ is then replaced by its complex conjugate which is the same as its transpose, since it is Hermitian for a loss free medium.

The theory in *R.w.i.* was treated in the context of the ionosphere regarded as a cold electron plasma. In that special case

$$\left.\begin{aligned} G = -XY^2/\{U(U^2-Y^2)\}, \quad D = -XY/(U^2-Y^2), \\ \tfrac{1}{2}(\epsilon_1+\epsilon_2) = 1-UX/(U^2-Y^2), \quad U = 1-\mathrm{i}Z, \end{aligned}\right\} \tag{4}$$

where X, Y, Z are the standard symbols of magnetoionic theory (Ratcliffe 1959; *R.w.i.* ch. 3). When the collision frequency ν is ignored, $Z = \nu/\omega$ is zero and U is unity.

It is now assumed that in one of the regions where ϵ_1, ϵ_2, ϵ_3 are constant, that is where $\pm s$ is real and large, there is an incident plane electromagnetic wave, whose field components all have an x and y dependence

$$\exp\{-\mathrm{i}k(S_1x+S_2y)\}, \tag{5}$$

where a time factor $\exp(+\mathrm{i}\omega t)$ is used. Then, because of Snell's law, the resulting disturbance has the same x and y dependence for all s. If the region of incidence is free space, S_1, S_2 are the x, y direction cosines of the wave normal there.

The theory of propagation in a stratified plasma is often formulated in terms of a 4×4 matrix $\boldsymbol{T}$ used by Clemmow & Heading (1954) (*R.w.i.* ch. 18) and this method is followed here. It can be shown (Walker & Lindsay 1975) that

$$\epsilon_{zz}\boldsymbol{T} = \begin{bmatrix} -\epsilon_{zx}S_1 & \epsilon_{zy}S_1 & S_1S_2 & \epsilon_{zz}-S_1^2 \\ \epsilon_{zx}S_2 & -\epsilon_{zy}S_2 & \epsilon_{zz}-S_2^2 & S_1S_2 \\ \epsilon_{yz}\epsilon_{zx}-\epsilon_{zz}(\epsilon_{yx}+S_1S_2) & -\epsilon_{yz}\epsilon_{zy}+\epsilon_{zz}(\epsilon_{yy}-S_1^2) & -\epsilon_{yz}S_2 & \epsilon_{yz}S_1 \\ -\epsilon_{xz}\epsilon_{zx}+\epsilon_{zz}(\epsilon_{xx}-S_2^2) & \epsilon_{xz}\epsilon_{zy}-\epsilon_{zz}(\epsilon_{xy}+S_1S_2) & \epsilon_{xz}S_2 & -\epsilon_{xz}S_1 \end{bmatrix}. \tag{6}$$

This expression may be more familiar for the special case $S_2 = 0$, as given by Budden (1972). Its derivation for the general case is straightforward.

Now (3) and (6) show that if the direction of the superimposed magnetic field is reversed, $\boldsymbol{\varepsilon}$ is transposed, and $\boldsymbol{T}$ is transposed about its trailing diagonal. The resulting matrix will be called the adjoint of $\boldsymbol{T}$ and written $\overline{\boldsymbol{T}}$. Thus

$$\overline{T}_{ij} = T_{5-j,\,5-i}. \tag{7}$$

(Budden & Clemmow 1957).

It is convenient to express this in matrix notation with the matrix $\boldsymbol{B}$ used by Bennett (1976):

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

with the properties

$$\boldsymbol{B}^{\mathrm{T}} = \boldsymbol{B}^{-1} = \boldsymbol{B}. \tag{9}$$

Then

$$\overline{\boldsymbol{T}} = \boldsymbol{B}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{B}, \tag{10}$$

where a superscript T denotes the transpose. The results (7)–(10) are still true when damping is included and when s is complex.

The elements on the right of (6) are all analytic functions of s, bounded when s is real. They may have poles where s is complex and these would be singularities of the differential equations (14) and (16) below. These poles might give rise to some interesting effects but their study is beyond the scope of this paper. Similarly the factor ϵ_{zz} on the left of (6) is bounded when s is real but in nearly all practical situations it has at least one zero for a real value of s, when damping is neglected. In the special case of magnetoionic theory with electrons only (3) and (4) show that this occurs where

$$\epsilon_{zz} = 1-X(1-n^2Y^2)/(1-Y^2) = 0, \tag{11}$$

which is just the condition that one of the refractive indices μ shall be infinite (Ratcliffe 1959). If $Y<1$, this is the infinity associated with the Z mode. If $n = 0$, (11) is the condition for the upper hybrid resonance. In a plasma with more than one ion species, other zeros of ϵ_{zz} can occur, associated with the lower hybrid resonances, and other resonances. In this paper we study the case where there is one isolated simple zero of ϵ_{zz} on or near the real s axis. It is a simple pole of each of the elements of $\boldsymbol{T}$ in (6) and, as shown below, it is a regular singularity of the governing differential equations. At least one of the solutions has an essential singularity there, and so it is necessary to introduce a branch cut, as will now be explained.

In the loss free case ϵ_{zz} is real when s is real, and it will now be assumed that it is a decreasing function of s at its zero. This would occur, for example, on the lower side of an ionospheric layer where the electron concentration, proportional to X, is an increasing function of s. Then it can be shown, see § 8, that in some important practical cases the barrier is in the region where Re (s) is less than its value at the zero of ϵ_{zz}, as in figure 1. If a small amount of collision damping is now introduced, (11) is replaced by

$$\epsilon_{zz} = 1 - X(U^2 - n^2Y^2)/\{U(U^2 - Y^2)\} = 0, \tag{12}$$

and it can be shown that, where (12) is satisfied, X, and therefore also s has a negative imaginary part. The cut must be drawn from this point to infinity without crossing the real s axis. It must therefore run from the zero of ϵ_{zz} to infinity in the lower half of the s plane. This must still apply in the limit when the damping is zero. The same singular point and the same cut are used for $\overline{\boldsymbol{T}}$.

If now a solution with a singularity is to be studied on a contour running from where s is real and positive to where it is real and negative, the contour cannot, in the loss free case, run along the real axis through the singularity. It must be indented on the positive imaginary side. It is now convenient to choose the origin of s at the zero of ϵ_{zz}. Then if $\arg s = 0$ when s is real and positive, we must take

$$\arg s = +\pi \text{ when } s \text{ is real and negative.} \tag{13}$$

This requirement has important consequences for the theory of § 6.

In the other case where ϵ_{zz}, near its zero, is an increasing function of s, the barrier would be on the side where Re (s) is greater, and the cut would have to be in the upper half of the s plane. This case could obviously be dealt with in a similar way to the example studied here.

3. The differential equations and their adjoint

The x and y components of the electric and magnetic fields of the waves in the plasma are written as a column matrix $\boldsymbol{e}$ with four elements E_x, $-E_y$, Z_0H_x, Z_0H_y, where Z_0 is the characteristic impedance of free space. Then Maxwell's equations and the constitutive relations of the medium together with (5) may be combined into a set of four first order differential equations which may be written in matrix form

$$\boldsymbol{e}' = -\mathrm{i}\boldsymbol{T}\boldsymbol{e} \tag{14}$$

(Clemmow & Heading 1954) where a prime ′ denotes $\partial/\partial s$.

These equations or variants of them are often used for computing ionospheric reflexion and transmission coefficients. The main applications are for low or very low frequencies when the damping is appreciable. The singularity of $\boldsymbol{T}$ is then well removed from the real height axis and gives no trouble. For other applications, however, when the damping is small, the singularity of $\boldsymbol{T}$ can be troublesome in computing. For the solution that is singular the effective wavelength in the medium is small near the singularity because of the large refractive index, and this means that a very small step size is needed in the integration routine, with a consequent very long computing time. The remedy is to integrate (14) along a contour in the complex s plane running into the upper half of the s plane well clear of the singularity. This technique has been successfully used by Smith (1974*b*, 1977).

Solutions of (14) are to be studied here in a domain of the complex s plane within which $\boldsymbol{T}$ has a simple pole at $s = 0$, and no other singularities.

We also use an adjoint column $\bar{\boldsymbol{e}}$ whose elements are the adjoint field components $\bar{E}_x$, $-\bar{E}_y$, $Z_0\bar{H}_x$, $Z_0\bar{H}_y$. These have been very fully studied by Suchy & Altman (1975). Their main properties are that they satisfy a similar set of equations to those satisfied by $\boldsymbol{e}$, but (*a*) for a fictitious 'adjoint' medium in which the direction of the superimposed magnetic field is reversed, and (*b*) the signs of all derivatives with respect to x, y, z are reversed. Thus their x, y dependence is not given by (5) but by

$$\exp\{+\mathrm{i}k(S_1 x + S_2 y)\}. \tag{15}$$

The result of (*a*) is that $\boldsymbol{T}$ is replaced by $\bar{\boldsymbol{T}}$, (7), (10). The result of (*b*) is that the sign on the right hand side of the differential equation is positive. Thus the adjoint differential equation is

$$\bar{\boldsymbol{e}}' = \mathrm{i}\bar{\boldsymbol{T}}\bar{\boldsymbol{e}}. \tag{16}$$

Now multiply (14) on the left by $\bar{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{B}$, multiply the transpose of (16) on the right by $\boldsymbol{Be}$, and add. This gives

$$(\partial/\partial s)(\bar{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{Be}) = \mathrm{i}(\bar{\boldsymbol{e}}^{\mathrm{T}}\bar{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{Be} - \bar{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{BTe}), \tag{17}$$

and (9) and (10) show that the right hand side is zero. Thus

$$W_z = \bar{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{Be} \tag{18}$$

is independent of s. This applies for all s, real or complex, that are not singularities of $\boldsymbol{e}$, $\bar{\boldsymbol{e}}$. It applies whether or not the effect of damping is included. The quantity (18) is the z component of the bilinear concomitant vector $\boldsymbol{W}$ (Suchy & Altman 1975).

The solution $\bar{\boldsymbol{e}}$ of (16) may be chosen in many different ways. Here the following choice will be made, again based on a suggestion of Suchy & Altman (1975). Consider a loss free medium where s is real. Here (3) and (6) show that $\bar{\boldsymbol{T}}$ is the complex conjugate $\boldsymbol{T}^*$ of $\boldsymbol{T}$. Then it follows, by taking the complex conjugate of (16), that $\boldsymbol{e}^*$ is a solution of (16) when $\boldsymbol{e}$ is a solution of (14). This applies only on the real s axis. Choose $\bar{\boldsymbol{e}}$ so that $\bar{\boldsymbol{e}} = \boldsymbol{e}^*$ where s is real and positive. Then (18) shows that

$$W_z = Z_0(E_x H_y^* + E_x^* H_y - E_y H_x^* - E_y^* H_x) = 4Z_0\Pi_z, \tag{19}$$

where Π_z is the z component of the time averaged Poynting vector. This is no longer true if we move into a region where the medium is absorbing or if we move off the real s axis. In both these cases $\bar{\boldsymbol{T}} \neq \boldsymbol{T}^*$.

Now suppose that the medium is loss free. It is required to follow the changes of $\boldsymbol{e}$, $\bar{\boldsymbol{e}}$, W_z and Π_z as we move along a contour from real positive s to real negative s. If the solutions $\boldsymbol{e}$, $\bar{\boldsymbol{e}}$ are singular at $s = 0$, the contour must be indented into the upper half s-plane to avoid the singularity. On the indentation W_z remains constant but Π_z changes and (19) does not remain true. When the real s axis is regained and Re (s) is negative, Π_z is again constant, but $\bar{\boldsymbol{e}} \neq \boldsymbol{e}^*$ and (19) is no longer true. The z-component of the energy flux is not the same on the positive and negative sides of the singularity or resonance. This result is used in § 6 to show that there must be an apparent disappearance of energy even in a loss free medium.

4. Solutions near the resonance

The matrix $\boldsymbol{T}$ in (6) and (14) can be expanded in a Laurent series with matrix coefficients, thus

$$\boldsymbol{T} = s^{-1}\boldsymbol{T}_{-1} + \boldsymbol{T}_0 + s\boldsymbol{T}_1 + s^2\boldsymbol{T}_2 + \dots. \tag{20}$$

In (6) the elements ϵ_{ij} are functions of s. Let

$$\{\partial\epsilon_{zz}(s)/\partial s\}_{s=0} = -1/A \tag{21}$$

so that in the loss free case A is real and positive. Then (6) shows that

$$\boldsymbol{T}_{-1} = \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} \tag{22}$$

where

$$\left.\begin{aligned}\boldsymbol{a}^{\mathrm{T}} &= A^{\frac{1}{2}}\{S_1, \quad -S_2, \quad -\epsilon_{yz}(0), \quad \epsilon_{xz}(0)\},\\ \boldsymbol{b}^{\mathrm{T}} &= A^{\frac{1}{2}}\{\epsilon_{zx}(0), \quad -\epsilon_{zy}(0), \quad -S_2, \quad S_1\}.\end{aligned}\right\} \tag{23}$$

It is no accident that $\boldsymbol{T}_{-1}$ has the simple outer product form (22). This occurs because E_z is eliminated from Maxwell's equations when (14) is derived. The contribution of E_z to $\partial\boldsymbol{e}^{\mathrm{T}}/\partial s$ in (14) is

$$-\mathrm{i}E_z(S_1, \quad -S_2, \quad -\epsilon_{yz}, \quad \epsilon_{xz}) \tag{24}$$

and the z component of the fourth Maxwell equation shows that

$$E_z = (\epsilon_{zz})^{-1}\{-\epsilon_{zx}E_x + \epsilon_{zy}(-E_y) + S_2Z_0H_x - S_1Z_0H_y\}. \tag{25}$$

The adjoint matrix $\overline{\boldsymbol{T}}$ can be expanded in a series similar to (20). It then follows that

$$\overline{\boldsymbol{T}}_{-1} = \bar{\boldsymbol{a}}\bar{\boldsymbol{b}}^{\mathrm{T}}, \tag{26}$$

where

$$\left.\begin{aligned}\bar{\boldsymbol{a}}^{\mathrm{T}} &= A^{\frac{1}{2}}\{S_1, \quad -S_2, \quad -\epsilon_{zy}(0), \quad \epsilon_{zx}(0)\} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{B},\\ \bar{\boldsymbol{b}}^{\mathrm{T}} &= A^{\frac{1}{2}}\{\epsilon_{xz}(0), \quad -\epsilon_{xy}(0), \quad -S_2, \quad S_1\} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{B}.\end{aligned}\right\} \tag{27}$$

The following properties are needed later:

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = \bar{\boldsymbol{b}}^{\mathrm{T}}\bar{\boldsymbol{a}} = \bar{\boldsymbol{a}}^{\mathrm{T}}\bar{\boldsymbol{b}} = \Delta, \tag{28}$$

where

$$\begin{aligned}\Delta &= A[S_1\{\epsilon_{xz}(0) + \epsilon_{zx}(0)\} + S_2\{\epsilon_{yz}(0) + \epsilon_{zy}(0)\}],\\ &= -2nAG(0)\,(lS_1 + mS_2).\end{aligned} \tag{29}$$

In a loss free system Δ is real if S_1, S_2 are real.

$$\bar{\boldsymbol{a}}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{a} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = \bar{\boldsymbol{b}}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{b} = \Delta, \tag{30}$$

$$\bar{\boldsymbol{a}}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} = \bar{\boldsymbol{a}}^{\mathrm{T}}\bar{\boldsymbol{a}} = \Omega \text{ (say)}, \tag{31}$$

$$\bar{\boldsymbol{b}}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{a} = \bar{\boldsymbol{b}}^{\mathrm{T}}\bar{\boldsymbol{b}} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = \Omega. \tag{32}$$

The equality of (31) and (32) can be proved by multiplying out with (3).

A solution of (14) can now be found by a standard method similar to that used by Ince (1927, ch. 15). We seek a series solution

$$\boldsymbol{e} = s^p(\boldsymbol{e}_0 + s\boldsymbol{e}_1 + s^2\boldsymbol{e}_2 + \dots). \tag{33}$$

Substitute in (14) and equate coefficients of successive powers of s. This gives

$$p\boldsymbol{e}_0 = \mathrm{i}\boldsymbol{T}_{-1}\boldsymbol{e}_0, \tag{34}$$

$$\left.\begin{aligned}(p+1+\mathrm{i}\boldsymbol{T}_{-1})\,\boldsymbol{e}_1 &= -\mathrm{i}\boldsymbol{T}_0\boldsymbol{e}_0,\\ (p+2+\mathrm{i}\boldsymbol{T}_{-1})\,\boldsymbol{e}_2 &= -\mathrm{i}(\boldsymbol{T}_0\boldsymbol{e}_1 + \boldsymbol{T}_1\boldsymbol{e}_0).\\ &\cdots\end{aligned}\right\} \tag{35}$$

Here (34) is the indicial equation. With (22) it shows that

either $$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{e}_0 = 0, \quad p = 0, \tag{36}$$

or $$\boldsymbol{e}_0 = \boldsymbol{a}, \quad p = -\mathrm{i}\Delta. \tag{37}$$

In the first equation of (37) it is implied that there is a constant factor on the right, equal to unity, with dimensions of electric field intensity. The vectors $\boldsymbol{e}$, $\boldsymbol{e}_i$ and their adjoints used below all have the dimensions of electric field intensity, whereas $\boldsymbol{T}$, $\boldsymbol{T}_i$, $\boldsymbol{a}$, $\boldsymbol{b}$, their adjoints, and Δ, Ω are all dimensionless. Now (33) shows that the solutions (36) are analytic at $s = 0$. The vector $\boldsymbol{e}_0$ must be perpendicular to $\boldsymbol{b}$ in four dimensional complex space. It can be chosen in three linearly independent ways. Thus there are three linearly independent solutions satisfying (36). There is only one solution (37) and this is linearly independent of the other three, provided $\Delta \neq 0$, and (33) shows that it then has an essential singularity at $s = 0$. Thus for $\Delta \neq 0$ there are four values of $\boldsymbol{e}_0$ giving independent solutions. Once $\boldsymbol{e}_0$ is known the remaining coefficients $\boldsymbol{e}_i$ in (33) can be found in succession from (35), for it can be shown, by using (22), that if r is any integer

$$\left.\begin{aligned}(r-\mathrm{i}\Delta+\mathrm{i}\boldsymbol{T}_{-1})^{-1} &= (r+\mathrm{i}\Delta)^{-1}(1-\mathrm{i}r^{-1}\boldsymbol{T}_{-1}),\\ (r+\mathrm{i}\boldsymbol{T}_{-1})^{-1} &= r^{-1}\{1-\mathrm{i}(r+\mathrm{i}\Delta)^{-1}\boldsymbol{T}_{-1}\}.\end{aligned}\right\} \tag{38}$$

When $\Delta = 0$, however, the solution (37) is not independent of the other three because now $\boldsymbol{e}_0 = \boldsymbol{a}$ satisfies the first equation (36). A way must be found of constructing a fourth, independent solution when $\Delta = 0$. The condition $\Delta = 0$ includes the case $S_1 = S_2 = 0$, from (29). This is the case of normally incident waves treated in previous work (*R.w.i.*). We need to find a solution, singular at $s = 0$, that is uniformly an independent fourth solution of (14) whether or not $\Delta = 0$. The solution (37) does not satisfy this requirement. A suitable solution is constructed in the following section.

The adjoint equation (16) can be treated in a similar way. We seek a series solution

$$\bar{\boldsymbol{e}} = s^{\bar{p}}(\bar{\boldsymbol{e}}_0 + s\bar{\boldsymbol{e}}_1 + s^2\bar{\boldsymbol{e}}_2 + \ldots). \tag{39}$$

On going through the steps analogous to (34) and (35) it is then found, on using (26)–(29) that

either $$\bar{\boldsymbol{b}}^{\mathrm{T}}\bar{\boldsymbol{e}}_0 = 0, \quad \bar{p} = 0 \tag{40}$$

giving three independent analytic solutions

or $$\bar{\boldsymbol{e}}_0 = \bar{\boldsymbol{a}} = \boldsymbol{B}\boldsymbol{b}, \quad \bar{p} = \mathrm{i}k\Delta = -p \tag{41}$$

giving one solution, singular at $s = 0$, if $\Delta \neq 0$.

Any solution of (14) that satisfies (36) is analytic at $s = 0$ and will be denoted by $\boldsymbol{e}_{\mathrm{A}}$, and its coefficients in (33) by $\boldsymbol{e}_{0\mathrm{A}}$, $\boldsymbol{e}_{1\mathrm{A}}$, etc. Similarly any solution that satisfies (37) is singular at $s = 0$ and will be denoted by $\boldsymbol{e}_{\mathrm{S}}$ with coefficients $\boldsymbol{e}_{0\mathrm{S}} = \boldsymbol{a}$, $\boldsymbol{e}_{1\mathrm{S}}$, etc. A corresponding notation is used for solutions of the adjoint equation (16), namely $\bar{\boldsymbol{e}}_{\mathrm{A}}$, analytic at $s = 0$, with coefficients $\bar{\boldsymbol{e}}_{0\mathrm{A}}$, $\bar{\boldsymbol{e}}_{1\mathrm{A}}$, etc. and $\bar{\boldsymbol{e}}_{\mathrm{S}}$, singular at $s = 0$, with coefficients $\bar{\boldsymbol{e}}_{\mathrm{S}0} = \bar{\boldsymbol{a}}$, $\bar{\boldsymbol{e}}_{1\mathrm{S}}$, etc.

Suppose now that we use a solution $\boldsymbol{e}_{\mathrm{A}}$, and an adjoint $\bar{\boldsymbol{e}}_{\mathrm{S}}$. The first terms of the series for these two solutions make a contribution to (18) given by

$$s^{\mathrm{i}\Delta}\bar{\boldsymbol{a}}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{e}_{0\mathrm{A}}. \tag{42}$$

But (27) and (36) show that this is zero. This and the contributions from other pairs of terms in the two series all vary with s. But W_z must be independent of s as shown in § 3, so that all these contributions must be zero. It can in fact be proved, by using (35) and (38) and their adjoints to find the coefficients $\boldsymbol{e}_{i\mathrm{A}}$, $\bar{\boldsymbol{e}}_{j\mathrm{S}}$, that each pair contributes zero to W_z. Next consider a solution $\boldsymbol{e}_\mathrm{S}$ and an adjoint $\bar{\boldsymbol{e}}_\mathrm{S}$. The resulting W_z is constant and therefore must be given by

$$\bar{\boldsymbol{e}}_{0\mathrm{S}}^\mathrm{T}\,\boldsymbol{B}\boldsymbol{e}_{0\mathrm{S}} = \bar{\boldsymbol{a}}^\mathrm{T}\boldsymbol{B}\boldsymbol{a} = \Delta \tag{43}$$

from (30), because this is the only term that does not depend on s. Again it can be checked by finding the coefficients $\bar{\boldsymbol{e}}_{i\mathrm{S}}$, $\boldsymbol{e}_{j\mathrm{S}}$ that all other products $\bar{\boldsymbol{e}}_{i\mathrm{S}}^\mathrm{T}\,\boldsymbol{B}\boldsymbol{e}_{j\mathrm{S}}$ are zero. Finally consider a solution $\boldsymbol{e}_\mathrm{A}$ and an adjoint $\bar{\boldsymbol{e}}_\mathrm{A}$. Their contribution to W_z is

$$W_z = \bar{\boldsymbol{e}}_{0\mathrm{A}}^\mathrm{T}\,\boldsymbol{B}\boldsymbol{e}_{0\mathrm{A}} \tag{44}$$

because all other terms in their series give zero. The value of (44) depends on the magnitudes of the vectors $\boldsymbol{e}_{0\mathrm{A}}$, $\bar{\boldsymbol{e}}_{0\mathrm{A}}$, which have not yet been defined.

5. Construction of solution with singularity

For one of the three analytic solutions choose a particular $\boldsymbol{e}_\mathrm{A}$ to be called $\boldsymbol{e}_\mathrm{a}$, for which

$$\boldsymbol{e}_{0\mathrm{a}} = \boldsymbol{a} - \Delta\boldsymbol{b}/\Omega. \tag{45}$$

Equations (28) and (31) show that this satisfies (36) as it must. Similarly for one of the three analytic adjoints take

$$\bar{\boldsymbol{e}}_{0\mathrm{a}} = \bar{\boldsymbol{a}} - \Delta\bar{\boldsymbol{b}}/\Omega \tag{46}$$

which satisfies (40). On the remaining two analytic solutions we can now impose the further condition that they are perpendicular to $\boldsymbol{a}$ as well as to $\boldsymbol{b}$, so that

$$\boldsymbol{e}_{0\mathrm{A}}^\mathrm{T}\,\boldsymbol{a} = \boldsymbol{e}_{0\mathrm{A}}^\mathrm{T}\,\boldsymbol{B}\bar{\boldsymbol{b}} = 0 \tag{47}$$

and similarly for the adjoints

$$\bar{\boldsymbol{e}}_{0\mathrm{A}}^\mathrm{T}\bar{\boldsymbol{a}} = \bar{\boldsymbol{e}}_{0\mathrm{A}}^\mathrm{T}\,\boldsymbol{B}\boldsymbol{b} = 0. \tag{48}$$

These ensure that if a solution $\boldsymbol{e}_{0\mathrm{A}}$ is used with an adjoint $\bar{\boldsymbol{e}}_\mathrm{a}$ or $\bar{\boldsymbol{e}}_\mathrm{S}$ their contributions to W_z are zero. Similarly solutions $\boldsymbol{e}_\mathrm{a}$, $\boldsymbol{e}_\mathrm{S}$ with an adjoint $\bar{\boldsymbol{e}}_{0\mathrm{A}}$ give no contribution to W_z. If a solution $\boldsymbol{e}_{0\mathrm{A}}$ or an adjoint $\bar{\boldsymbol{e}}_{0\mathrm{A}}$ is present, they have no singularities at $s = 0$. They do not interact with $\boldsymbol{e}_\mathrm{a}$, $\boldsymbol{e}_\mathrm{S}$, $\bar{\boldsymbol{e}}_\mathrm{a}$, $\bar{\boldsymbol{e}}_\mathrm{S}$ to give contributions to W_z and we may say that they are 'independently propagated' near $s = 0$. They need not be considered further here.

Note, next, that the contribution of (45) and (46) to W_z is

$$\bar{\boldsymbol{a}}^\mathrm{T}\boldsymbol{B}\boldsymbol{a} + (\Delta^2/\Omega^2)\,\bar{\boldsymbol{b}}^\mathrm{T}\boldsymbol{B}\boldsymbol{b} - (\Delta/\Omega)\,(\bar{\boldsymbol{a}}^\mathrm{T}\boldsymbol{B}\boldsymbol{b} + \bar{\boldsymbol{b}}^\mathrm{T}\boldsymbol{B}\boldsymbol{a}) = -\Delta + \Delta^3/\Omega^2 \tag{49}$$

where (30)–(32) have been used.

Now take as the required fourth solution

$$\boldsymbol{e}_z = (\boldsymbol{e}_\mathrm{S} - \boldsymbol{e}_\mathrm{a})/\Delta. \tag{50}$$

We may write this

$$\boldsymbol{e}_z = \{s^{-\mathrm{i}\Delta}\boldsymbol{e}_{\mathrm{aS}}(\Delta) - \boldsymbol{e}_\mathrm{a}(\Delta)\}/\Delta \tag{51}$$

where $\boldsymbol{e}_{\mathrm{aS}}(\Delta)$ and $\boldsymbol{e}_\mathrm{a}(\Delta)$ are the series in (39). They are different functions of Δ but tend to the same limit $\boldsymbol{e}_\mathrm{a}$ when $\Delta = 0$. Then it follows that

$$\lim_{\Delta\to 0} \boldsymbol{e}_z = -\mathrm{i}(\ln s)\,\boldsymbol{e}_\mathrm{a} + \frac{\mathrm{d}}{\mathrm{d}\Delta}\{\boldsymbol{e}_{\mathrm{aS}}(\Delta) - \boldsymbol{e}_\mathrm{a}(\Delta)\}_{\Delta=0}. \tag{52}$$

Thus (50) tends to a non-zero bounded limit as $\Delta \to 0$. It is obviously singular when $s = 0$, and it is linearly independent of the other three, analytic, solutions $\boldsymbol{e}_\mathrm{a}$ and the two $\boldsymbol{e}_\mathrm{A}$'s. It therefore satisfies the requirement of § 4 that it is uniformly an independent solution, singular at $s = 0$, whether or not $\Delta = 0$.

The above process is very similar to that used for constructing a second solution $Y_\nu(z)$ of Bessel's equation when the order ν tends to an integer (see for example Watson 1944, §§ 3.5–3.53). An illustration using Bessel functions is given in § 7.

Similarly take as the corresponding fourth solution of the adjoint equation

$$\bar{\boldsymbol{e}}_z = (\bar{\boldsymbol{e}}_S - \bar{\boldsymbol{e}}_a)/\Delta. \tag{53}$$

Then the contribution of (50) (53) to W_z can be found from (43) (49) and is

$$\Delta^{-2}\{\bar{\boldsymbol{e}}_S^T \boldsymbol{B}\boldsymbol{e}_S + \bar{\boldsymbol{e}}_a^T \boldsymbol{B}\boldsymbol{e}_a\} = \Delta/\Omega^2. \tag{54}$$

Thus it is zero when $\Delta = 0$. This is because (50) and (53) resemble perfect standing waves. The solution $\boldsymbol{e}_{0a}$ of (45) and its adjoint (46) also give $W_z = 0$ when $\Delta = 0$, as shown by (49). Thus these solutions also resemble perfect standing waves when $\Delta = 0$. To get a solution representing a travelling wave we must take a linear combination of $\boldsymbol{e}_z$, $\boldsymbol{e}_a$ and of their adjoints. Thus consider a general solution

$$\boldsymbol{e} = P\boldsymbol{e}_a + Q\boldsymbol{e}_z \tag{55}$$

and an adjoint

$$\bar{\boldsymbol{e}} = P^*\bar{\boldsymbol{e}}_a + Q^*\bar{\boldsymbol{e}}_z. \tag{56}$$

When s is real and positive these are complex conjugates and so $W_z = 4Z_0\Pi_z$ from (19). But (43), (49), (54) show that

$$W_z = PP^*(-\Delta + \Delta^3/\Omega^2) + QQ^*\Delta/\Omega^2 + (PQ^* + P^*Q)(1 - \Delta^2/\Omega^2). \tag{57}$$

If $\Delta = 0$ this gives

$$W_z = PQ^* + P^*Q \tag{58}$$

which is in general non-zero when $\Delta = 0$ and shows that (55) and (56) represent a perfect or a partial progressive wave.

The values of P, Q needed to give a perfect progressive wave can only be settled by examining the solution (55) in a region of the s plane where the medium is homogeneous or sufficiently slowly varying for some form of asymptotic or W.K.B. solution to be used to define a progressive wave. This aspect of the problem is taken up in § 8.

6. Loss of energy at the singularity

Consider again the solution (55) and its adjoint (56), which are complex conjugates when s is real and positive so that there the upward energy flux Π_z is $\frac{1}{4}Z_0$ times (57), from (19). Now (55) includes a term $Q\boldsymbol{e}_S/\Delta$ from (50) which has a factor $s^{-i\Delta}$. The first term of the series (33) for $\boldsymbol{e}_S$ therefore contributes

$$Qs^{-i\Delta}\boldsymbol{a}/\Delta, \tag{59}$$

and this is the only term of the series that contributes to W_z, (57) as shown in § 4. Similarly the adjoint (56) contains a corresponding term

$$Q^*s^{i\Delta}\bar{\boldsymbol{a}}/\Delta. \tag{60}$$

The terms (59) and (60), which are singular at $s = 0$, give a contribution to W_z equal to QQ^*/Δ from (30). It does not appear in (57) because it is cancelled by a contribution from the term $-\boldsymbol{e}_a/\Delta$ in (50), and its adjoint, which are analytic at $s = 0$. This cancelling contribution comes from the term $-\Delta$ in (49). Now we move to where s is real and negative. This must be done by going along a contour indented into the upper half s plane, to avoid the singularity. According to (13) $\arg s$ increases from 0 to $+\pi$ so $s^{-i\Delta}$ in (59) becomes $|s|^{-i\Delta}\,e^{\pi\Delta}$ and $s^{i\Delta}$ in (60) becomes

$|s|^{i\Delta}\,e^{-\pi\Delta}$. The product of these two factors is unchanged and their contribution to W_z remains the same as before, and is still cancelled. But (59), (60) and therefore (55), (56) are no longer complex conjugates. Then (19) is no longer true.

To find Π_z when s is real and negative we must follow the function (60) along a contour indented into the lower half s plane so that it remains the complex conjugate of (59). Then $s^{i\Delta}$ in (60) becomes $|s|^{i\Delta}\,e^{\pi\Delta}$. Both (59) and (60) have been multiplied by $e^{\pi\Delta}$. Their contribution to $4Z_0\Pi_z$ in (19) is thus multiplied by $e^{2\pi\Delta}$ and is no longer cancelled. Now (59) and (60) are the only terms with singularities that contribute to W_z or Π_z and therefore they are the only terms to be affected by indentation of the contour. Hence, if we write $\Pi_z(-)$ and $\Pi_z(+)$ for the z component of the Poynting flux when Re (s) is negative and positive, respectively, then

$$4Z_0\{\Pi_z(-)-\Pi_z(+)\} = QQ^*(e^{2\pi\Delta}-1)/\Delta = QQ^*(2\pi+2\pi^2\Delta+\ldots), \tag{61}$$

where (30) has been used. This is positive, whatever the sign of Δ. It shows that there must always be a disappearance of energy, in the loss free case at a resonance, if the solution with a singularity is present.

This proof has been given for the case $\Delta \neq 0$, but the result (61) is still true in the limit $\Delta \to 0$. This could be established by constructing a separate proof for the case $\Delta = 0$. It would use for $\boldsymbol{e}_z$ the solution (52), containing $\ln s$. It need not be given here because it is done for a specific example in § 7. The result is the same as (61) with $\Delta = 0$.

In the special case where $P = 0$, $\Delta = 0$ in (55), (56) it follows from (19), (57) that $\Pi_z(+) = 0$, so that (61) gives

$$\Pi_z(-) = \tfrac{1}{2}\pi QQ^*/Z_0 \tag{62}$$

and the whole of this energy flux, coming in from where s is negative, disappears at $s = 0$.

As another example suppose that $P = -\pi Q$, $\Delta = 0$. Then (19), (57), (61) give $\Pi_z(+) = -\tfrac{1}{2}\pi QQ^*/Z_0$, $\Pi_z(-) = 0$. Here the energy flux comes in from where s is positive and all disappears at $s = 0$. The need for writing $-P/Q = \pi$ in this example seems to be similar to the need for a factor $1/\pi$ in Weber's definition of a Bessel function of the second kind (Watson 1944, p. 64).

It must be stressed that the energy fluxes in the solutions that are analytic at $s = 0$ are unaffected by the resonance. In this rather artificial example where the medium is assumed to be completely loss free, the energy that is lost from the solution with the singularity really does disappear. For example, Chessell (1971), in calculations for ionospheric layers, found that as much as 65 % of the incident energy was lost at a pole of the refractive index, even when electron collisions are reduced to zero. Emphatically this lost energy does not reappear, through a process of mode conversion, in other modes of propagation, as has sometimes been suggested. The question as to what happens to the lost energy is discussed in § 13.

7. An illustrative example: ionosphere at the magnetic equator

The foregoing results may be illustrated by an example in which the waves are normally incident on the plasma, so $S_1 = S_2 = 0$, and the superimposed magnetic field is parallel to the x axis, so $m = n = 0$, $l = 1$. It is further assumed that the plasma is a loss free cold electron plasma, so that the formulae of magnetoionic theory (Ratcliffe 1959) may be used. This is close to the situation that occurs when radio waves are vertically incident from below on the ionosphere at the magnetic equator. Now all except four of the elements of $\boldsymbol{T}$, (6), are zero and the equation (14) separates into two sets

$$\begin{bmatrix} E_x \\ Z_0 H_y \end{bmatrix}' = -\mathrm{i}\begin{bmatrix} 0 & 1 \\ \mu_o^2 & 0 \end{bmatrix}\begin{bmatrix} E_x \\ Z_0 H_y \end{bmatrix}, \tag{63}$$

and

$$\begin{bmatrix} -E_y \\ Z_0 H_x \end{bmatrix}' = -\mathrm{i}\begin{bmatrix} 0 & 1 \\ \mu_e^2 & 0 \end{bmatrix}\begin{bmatrix} -E_y \\ Z_0 H_x \end{bmatrix}. \tag{64}$$

Here $\mu_o^2 = \epsilon_3$, and $\mu_e^2 = \epsilon_1 \epsilon_2/\epsilon_{zz}$ are the squared refractive indices for the Ordinary and Extraordinary waves respectively. It is only μ_e^2 that shows a resonance, in this case the upper hybrid resonance where $X = 1 - Y^2$, and so we consider only (64), and take $E_x = H_y = 0$.

Suppose now that

$$\mu_e^2 = \beta/s. \tag{65}$$

This example was studied in *R.w.i.* (§ 21.14). Then (64) and its adjoint are

$$\begin{bmatrix} -E_y \\ Z_0 H_x \end{bmatrix}' = -\mathrm{i}\begin{bmatrix} 0 & 1 \\ \beta/s & 0 \end{bmatrix}\begin{bmatrix} -E_y \\ Z_0 H_x \end{bmatrix}, \quad \begin{bmatrix} -\bar{E}_y \\ Z_0 \bar{H}_x \end{bmatrix}' = \mathrm{i}\begin{bmatrix} 0 & 1 \\ \beta/s & 0 \end{bmatrix}\begin{bmatrix} -\bar{E}_y \\ Z_0 \bar{H}_x \end{bmatrix} \tag{66}$$

which show that E_y and $\bar{E}_y$ satisfy the same differential equation

$$\frac{\mathrm{d}^2 E_y}{\mathrm{d}s^2} + \frac{\beta}{s} E_y = 0. \tag{67}$$

Now from (18)

$$W_z = -Z_0(\bar{H}_x E_y + H_x \bar{E}_y) = \mathrm{i}\left(\bar{E}_y \frac{\partial E_y}{\partial s} - E_y \frac{\partial \bar{E}_y}{\partial s}\right). \tag{68}$$

The last bracket is the Wronskian. It was proved in § 3 that W_z must be constant. In this example it follows since any Wronskian of (67) is constant because there is no first derivative term.

Equation (67) was derived in *R.w.i.* from Försterling's (1942) coupled equations by neglecting the coupling terms and the coupling parameter ψ. This left open the possibility that any lost energy might have been accounted for through mode conversion via the neglected terms. In the present example, however, there are no approximations of this kind. With the condition $n = 0$ used here, Försterling's ψ is zero. The Försterling variable $\mathscr{F}$ is exactly proportional to E_y, and equations (66), (67) include no approximations.

Now (67) has the general solution

$$E_y, \quad \bar{E}_y = s^{\frac{1}{2}} \mathscr{C}_1\{2(\beta s)^{\frac{1}{2}}\}, \tag{69}$$

(Watson 1944 p. 97; *R.w.i.* p. 475) where $\mathscr{C}$ denotes any Bessel function. We choose E_y and $\bar{E}_y$ so that they are complex conjugates when s is real and positive, as explained in § 3. We could choose $\mathscr{C} = J$ for both, and (68) shows that this would give $W_z = 0$. The choice $\mathscr{C} = Y$ (Bessel function of the second kind) would similarly give $W_z = 0$. This is because, when s is real and positive, the functions J, Y, represent perfect standing waves with no total energy flux.

Consider the general solution

$$\left.\begin{aligned} E_y &= s^{\frac{1}{2}}[PJ_1\{2(\beta s)^{\frac{1}{2}}\} + \pi Q Y_1\{2(\beta s)^{\frac{1}{2}}\}], \\ \bar{E}_y &= s^{\frac{1}{2}}[P^* J_1\{2(\beta s)^{\frac{1}{2}}\} + \pi Q^* Y_1\{2(\beta s)^{\frac{1}{2}}\}]. \end{aligned}\right\} \tag{70}$$

This gives

$$W_z = \mathrm{i}(P^* Q - P Q^*) \tag{71}$$

as can be shown by using the formula for the Wronskian given by Watson (1944, p. 76). The functions in (70) and their derivatives can be expanded in series thus (Watson 1944, p. 62):

$$s^{\frac{1}{2}} J_1\{2\beta s)^{\frac{1}{2}}\} = \beta^{\frac{1}{2}} s - \tfrac{1}{2}\beta^{\frac{3}{2}} s^2 + \ldots, \tag{72}$$

$$s^{\frac{1}{2}}Y_1\{2(\beta s)^{\frac{1}{2}}\} = -\beta^{-\frac{1}{2}}\pi^{-1} + \pi^{-1}(\ln s)\{\beta^{\frac{1}{2}}s - \tfrac{1}{2}\beta^{\frac{3}{2}}s^2 + \dots\} + a_1 s + a_2 s^2 + \dots, \tag{73}$$

$$\frac{\partial}{\partial s}[s^{\frac{1}{2}}J_1\{2(\beta s)^{\frac{1}{2}}\}] = \beta^{\frac{1}{2}} - \beta^{\frac{3}{2}}s + \dots, \tag{74}$$

$$\frac{\partial}{\partial s}[s^{\frac{1}{2}}Y_1\{2(\beta s)^{\frac{1}{2}}\}] = \frac{\beta^{\frac{1}{2}}}{\pi}(\ln s)\{1 - \beta s + \dots\} + a_0' + a_1' s + \dots. \tag{75}$$

The values of the constants a_i, a_i' are not needed here. These series provide an alternative way of finding W_z. The only terms that contribute are those whose product is independent of s, that is $-1/\pi\beta^{\frac{1}{2}}$ in (73) and $\beta^{\frac{1}{2}}$ in (74). This confirms the result (71). There are two other terms, from a_0' in (75) and $-1/\pi\beta^{\frac{1}{2}}$ in (73), that cancel. These come from

$$\left.\begin{aligned} &\mathrm{i}\pi^2 Q^* s^{\frac{1}{2}}Y_1\{2(\beta s)^{\frac{1}{2}}\} \times \frac{\partial}{\partial s}[Q s^{\frac{1}{2}}Y_1\{2(\beta s)^{\frac{1}{2}}\}], \\ \text{and}\qquad &-\mathrm{i}\pi^2 \frac{\partial}{\partial s}[Q^* s^{\frac{1}{2}}Y_1\{2(\beta s)^{\frac{1}{2}}\}] \times Q s^{\frac{1}{2}}Y_1\{2(\beta s)^{\frac{1}{2}}\}. \end{aligned}\right\} \tag{76}$$

The terms on the left are from the adjoint and those on the right from the solution.

The solution (70) and its adjoint are complex conjugates when s is real and positive, and (71), (19) show that

$$\Pi_z = \tfrac{1}{4}\mathrm{i}(P^*Q - PQ^*)/Z_0. \tag{77}$$

We now move to where s is real and negative. This must be done on a contour indented into the upper half s plane to avoid the singularity at the origin. The functions (70) are analytic and W_z remains unchanged. In (75) the term $(\beta^{\frac{1}{2}}/\pi)\ln s$ becomes $(\beta^{\frac{1}{2}}/\pi)\ln|s| + \mathrm{i}\beta^{\frac{1}{2}}$, so that there is an extra term πQQ^* in the top line of (76). It is cancelled by an equal term from the bottom line.

To find Π_z when s is real and negative, however, the adjoint must be replaced by a function that remains the complex conjugate when the transition is made. Thus for the terms on the left of (76), which came from the adjoint, the path must be a contour indented on the negative imaginary side of the origin. Then in (75) $(\beta^{\frac{1}{2}}/\pi)\ln s$ becomes $(\beta^{\frac{1}{2}}/\pi)\ln|s| - \mathrm{i}\beta^{\frac{1}{2}}$. Both top and bottom lines of (76) now contribute extra terms πQQ^* that add together. This shows that

$$\Pi_z(-) - \Pi_z(+) = \tfrac{1}{2}\pi QQ^*/Z_0 \tag{78}$$

which is the same as (61) with $\Delta = 0$, derived for the more general case.

In *R.w.i.*, p. 475 a solution (69) was studied with $\mathscr{C}$ a Hankel function $H_1^{(1)}$. This represents an incident wave coming down from where s is positive, and it was shown that there is no reflected wave. This solution requires $P = 1$, $Q = \mathrm{i}/\pi$ in (70) (Watson 1944, p. 74). Thus from (19), (71) $\Pi_z(+) = -\frac{1}{2}\pi Z_0$ and hence from (78), $\Pi_z(-) = 0$. The downcoming energy apparently disappears.

A similar analysis could be done by replacing (65) by $\mu_e^2 = \gamma + \beta/s$, where γ is a constant, so that μ_e^2 has a zero where $s = -\beta/\gamma$ and there is a barrier where $-\beta/\gamma < s < 0$. This case was studied in *R.w.i.*, § 21.15. The equations that replace (66), (67) are still exact because no coupling terms are neglected. Solutions were given in terms of Whittaker functions as used in § 12, and it was shown that some energy apparently disappears. Again it cannot be accounted for by mode coupling because in this example there is none.

8. The characteristic waves, or modes

In order to give meaning to the terms 'barrier' and 'tunnelling', it is necessary to examine the the characteristic waves, or modes. For a loss free system two of these characteristic waves are attenuated within a barrier, and outside the barrier they are propagated. In a medium to which (3), (5), (6) apply there are four characteristic waves. If the medium were homogeneous, (3) and (6) would be independent of z, that is of s, and the s dependence of all field components in one characteristic wave would be given by a factor

$$\exp(-\mathrm{i}kqz) = \exp(-\mathrm{i}qs). \tag{79}$$

Then (14) shows that the four values of q are given by the Booker (1939) quartic equation

$$\det(\boldsymbol{T}-q\mathbf{1}) = 0, \tag{80}$$

which may be written

$$\epsilon_{zz}q^4 + a_3q^3 + a_2q^2 + a_1q + a_0 = 0. \tag{81}$$

By multiplying out the determinant in (80) it can be shown that

$$\left.\begin{aligned} a_3 &= -2Gn(S_1l+S_2m),\\ a_1 &= -2n(S_1l+S_2m)\{\tfrac{1}{2}\epsilon_3(\epsilon_1+\epsilon_2)-\epsilon_1\epsilon_2+G(S_1^2+S_2^2)\},\end{aligned}\right\} \tag{82}$$

and more complicated expressions for a_2, a_0, not needed here.

If the medium varies slowly enough with s, the idea of characteristic waves can still be used for most values of s. The factor (79) is now replaced by

$$\exp\left(-\mathrm{i}\int^s q\,\mathrm{d}s\right), \tag{83}$$

and the corresponding solutions are called the W.K.B. solutions. It is thus important to study how q depends on s.

The W.K.B. solutions with good approximation represent waves that are independently propagated provided that the four roots of (81) are distinct. The W.K.B. solutions fail near points in the complex s-plane where two roots of (81) are equal, and these are called 'coupling points'. They include 'reflexion points', and are sometimes called 'turning points'. The two characteristic waves associated with these two roots lose their separate identity in a domain surrounding a coupling point.

The series solutions $\boldsymbol{e}_{\mathrm{A}}$, $\boldsymbol{e}_{\mathrm{S}}$, (33), (39) used earlier are not necessarily characteristic waves. Their radius of convergence is limited only by singularities of the differential equation (14). The coupling points are not singularities but ordinary points (Budden 1972). Thus any series solution might be a mixture of characteristic waves, in proportions that are nearly constant at points remote from coupling points, but vary rapidly near coupling points. This is why it is necessary to make a separate examination of the characteristic waves.

For normal incidence, $S_1 = S_2 = 0$, (82) shows that $a_1 = a_3 = 0$ so that (81) is a quadratic for q^2. Its four roots are now denoted by $\pm\mu_{\mathrm{o}}$, $\pm\mu_{\mathrm{e}}$ where μ_{o}, μ_{e} are the refractive indices for the Ordinary and Extraordinary waves respectively. In this case there is a reflexion point where either μ_{o} or μ_{e} is zero.

In the most familiar form of barrier the square of one of the refractive indices, say μ_{e}^2, is negative for some range of the real s axis. At the ends of this range μ_{e}^2 is zero and these boundaries of the barrier are reflexion points. Outside the barrier, the two roots $\pm\mu_{\mathrm{e}}$ are real and correspond to two waves propagating in opposite directions. But one value of μ^2 is infinite where ϵ_{zz} is zero, and this infinity can form one boundary of a barrier. It can be shown that, for a cold electron plasma, the coefficient a_2 in (81) must be negative at $s = 0$ where $\epsilon_{zz} = 0$. This applies, for example, to the ionosphere. Now for small $|s|$, one value of μ^2 is given by

$$\mu^2 \approx -a_2/\epsilon_{zz} \tag{84}$$

and since we are assuming (see § 2) that ϵ_{zz}, near its zero, is a decreasing function of s, it follows that μ^2 is negative when s is negative. Thus the barrier is on the side of the resonance where s is negative. It will here be assumed that this is true. Cases where a_2 has the opposite sign can easily be dealt with by similar methods.

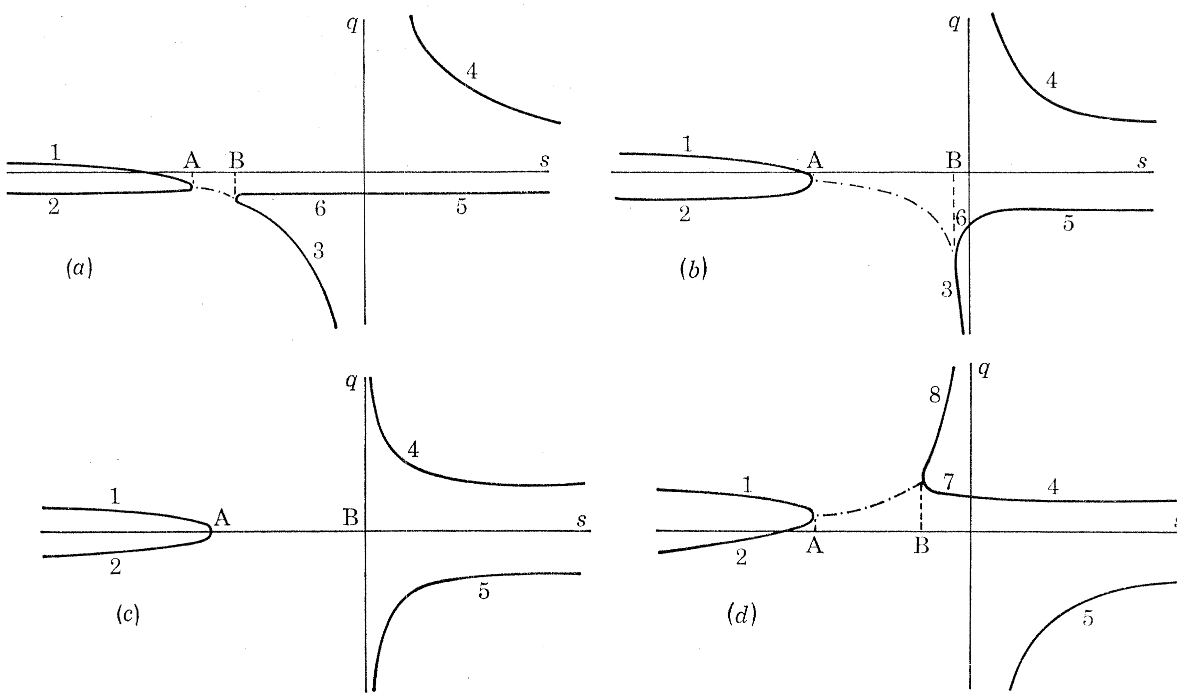

FIGURE 1. Some examples of how q may depend on $s = kz$ in and near a barrier. The chain curves show Re (q_1) = Re (q_2) in the barrier region where $(q_1-q_2)^2$ is negative. Figure 1*a* is for Δ large and positive and is typical of very oblique incidence. Two roots q are equal at A and at B, and between them the barrier is similar to a conventional barrier. Figure 1*b* is for nearly normal incidence when Δ is small and positive. The coupling point at B is still where the difference of the q's is zero, but its properties are modified by the proximity of the resonance. Figure 1*c* is the degenerate case of normal incidence with $\Delta = 0$. The coupling point B has moved to coincidence with the resonance, which therefore forms one end of the barrier. Figure 1*d* is for a negative value of Δ. The coupling point B is again where s is negative.

For oblique incidence and a loss free system the coefficients in (81) are all real when s is real, and Booker (1939) pointed out that then the roots q are either (*a*) all real, or (*b*) two are real and two are a complex conjugate pair, or (*c*) there are two complex conjugate pairs. Suppose that there is a coupling point for two roots, say q_1, q_2 on the real axis. Then $(q_1-q_2)^2$ is positive on one side of it, and here the two waves are propagated in opposite directions. On the other side of it $(q_1-q_2)^2$ is negative and here the two waves are attenuated in opposite senses. A barrier is a range of the real s axis where $(q_1-q_2)^2$ is negative, and it is zero at the ends. An example is the range AB in figure 1*a*. Thus for oblique incidence q_1-q_2 plays the same role as does the refractive index μ for normal incidence. It is in this sense that the term 'effective refractive index' is used for $q_1 - q_2$ in the abstract and introduction. This paper is concerned only with coupling points that are on the real s-axis, and are transitions between the regimes (*a*), (*b*), (*c*). There are other coupling points not on the real axis that are important (see, for example, *R.w.i.*) but are not used here.

A singularity of the differential equation (14) occurs at $s = 0$ where $\epsilon_{zz} = 0$, and (81) shows that one of its four roots is infinite there. If $a_3 \neq 0$, only one root is infinite and if $|s|$ is small enough this is given by

$$q \approx -a_3/\epsilon_{zz} \propto 1/s. \tag{85}$$

Thus on both sides of the singularity q is real and the associated waves are propagated. A resonance where only one q is infinite cannot be inside a barrier and cannot be the boundary of a barrier. See figure 1. (Other examples can be seen in curves given for example, by Booker 1939, Smith 1974*a*, *R.w.i.*) For normal incidence, however, (82) shows that $a_3 = 0$ and (29) shows that then $\Delta = 0$. Now two roots of the quartic (81) are equal at $s = 0$, because a coupling point, B in figure 1, has moved to coincidence with the singularity or resonance. It is for this reason that the barriers studied in *R.w.i.* were degenerate, and Stix (1962) refers to this kind of resonance as a 'singular turning point'. In the following sections an attempt is made to disentangle the separate effects of the coupling point and the resonance.

Other cases of resonance can occur. For example it can happen that in (81) $a_3 = a_1 \equiv 0$, and a_2, a_0 both contain a factor ϵ_{zz}. Then a kind of resonance can occur within a barrier region. A case of this kind was considered in *R.w.i.*, §§ 16.12–16.17 and by Hirsch & Shmoys (1965) but is not studied further here.

Figure 1 illustrates the problem. For a loss free system there is a resonance on the real height axis at $s = 0$, and reflexion points at A, B. In figure 1*a* there could be an incident upgoing wave where s is negative, branch 1 of the curves. The results of § 12 below show that part of its energy is reflected at A to give a downgoing wave, branch 2, but some tunnels through to give an upgoing wave above B, branch 3. This then encounters the resonance and some energy must disappear, but there can be a residual upgoing transmitted wave where s is positive, branch 4. Figure 1*d* is similar but the tunnelling wave can attain branch 7 and thence branch 4 without encountering the resonance. If the angle of incidence is small, however, the resonance is near B, the downgoing wave of branch 8 loses its physical identity, and there can be a loss of some energy.

Alternatively there could be a downgoing incident wave where s is positive, branch 5. In figure 1*a* this goes to branch 6 and is partly reflected at B, but some energy tunnels through to give a downgoing wave where s is negative, branch 2. It is shown in § 12 that there is no outgoing reflected wave, branch 4, and this must be because the reflected energy, branch 3, is lost at the resonance. In figure 1*d* the downgoing incident wave, branch 5, first encounters the resonance and some of its energy must disappear. The remainder tunnels through to give a downgoing wave below the barrier, branch 2. It is shown in § 12 that again there is no outgoing reflected wave, branch 4, and this must be because the properties of the reflecting system, the reflexion point at B with the resonance near it, are so modified that there is no reflexion.

The reasons why branch 3, figure 1*a*, is associated with an upgoing wave, and branch 8, figure 1*d*, with a downgoing wave are given later, § 10.

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9. Coupled equations

The W.K.B. solutions for the characteristic waves are conveniently studied by using the coupled equations (87) of Clemmow & Heading (1954). Let $\boldsymbol{s}_i$ be an eigen column of $\boldsymbol{T}$ with eigen value q_i and let $\boldsymbol{S}$ be the 4×4 matrix of the four columns $\boldsymbol{s}_i$. Define four new dependent variables, as a column $\boldsymbol{f}$ satisfying

$$\boldsymbol{e} = \boldsymbol{S}\boldsymbol{f}. \tag{86}$$

Then $\boldsymbol{f}$ satisfies

$$\boldsymbol{f}' = -\mathrm{i}\boldsymbol{Q}\boldsymbol{f} + \boldsymbol{\Gamma}\boldsymbol{f}, \quad \boldsymbol{\Gamma} = -\boldsymbol{S}^{-1}\boldsymbol{S}', \tag{87}$$

where $\boldsymbol{Q}$ is the diagonal matrix whose elements are q_i. The terms $\boldsymbol{\Gamma f}$ are called coupling terms and $\boldsymbol{\Gamma}$ is called the coupling matrix. All its elements depend on derivatives with respect to s of the $\boldsymbol{s}_i$. But the $\boldsymbol{s}_i$ are constant in a homogeneous medium, and their derivatives are small in a slowly varying medium. Thus $\boldsymbol{\Gamma}$ is small in a slowly varying medium provided $\boldsymbol{S}$ is non-singular. If the coupling terms are neglected in (87), the remaining equations separate into four first order differential equations for the f_i, whose solutions are the W.K.B. solutions. But a coupling point is where two q_i's and thence two $\boldsymbol{s}_i$'s are equal so that $\boldsymbol{S}$ is singular. Thus (86), (87) cannot be used near a coupling point and an alternative to (87) is needed, given below ((104)).

Coupled equations such as (87) or (104) are not really intended for accurate computing, although (87) has been successfully used by Arantes & Scarabucci (1975). Their purpose is to enable physical processes such as coupling, reflexion and resonance to be studied. They are often used with approximations, such as the neglect of the coupling terms. This could lead to errors but these are small in the practical cases that have been studied by a comparison with computed solutions. There do not seem to be any examples of physical importance where the errors are serious. For computing it is easier to use the original equations (14) or variants of them.

Before studying the neighbourhood of a coupling point, we must examine whether (87) can be used near a resonance. The columns $\boldsymbol{s}_i$ in $\boldsymbol{S}$ are eigen columns of the matrix on the right of (6). On the real s axis, including $s = 0$, all elements of this matrix are bounded, and so its eigen columns are bounded. Provided $a_3 \neq 0$ at $s = 0$, that is $\Delta \neq 0$, the quartic (81) has only one infinite root there. If the other three roots are distinct then it follows that the four $\boldsymbol{s}_i$ are distinct. Each $\boldsymbol{s}_i$ may be multiplied by an arbitrary 'normalizing' factor. It is possible to choose these factors so that $\boldsymbol{S}$ is bounded and non-singular at $s = 0$. Thus the coupled equation (87) can be used for a slowly varying medium in a domain of the s plane containing a resonance, provided that there are no coupling points in or near the domain. This means that $|\Delta|$, in (92) below, must not be small. The branch cut of § 2 must, of course, be used. For a cold electron plasma, a form of $\boldsymbol{S}$ suitable for this was given by Budden & Clemmow (1957), (and also *R.w.i.* (18.74)), and it can be checked that the elements of their $\boldsymbol{S}$ are all bounded where $\varepsilon_{zz} = 0$.

Similar conclusions apply for the adjoint system. The quartic (81) is the same. The adjoint of (86) is

$$\bar{\boldsymbol{e}} = \bar{\boldsymbol{S}}\bar{\boldsymbol{f}} \tag{88}$$

and it was shown by Budden & Clemmow (1957) that, with their normalization of $\boldsymbol{S}$,

$$\bar{\boldsymbol{S}}^{\mathrm{T}} = \boldsymbol{S}^{-1}\boldsymbol{B} \tag{89}$$

and the adjoint coupled equations are

$$\bar{\boldsymbol{f}}' = \mathrm{i}\boldsymbol{Q}\bar{\boldsymbol{f}} + \bar{\boldsymbol{\Gamma}}\bar{\boldsymbol{f}}, \quad \bar{\boldsymbol{\Gamma}} = -\bar{\boldsymbol{S}}^{-1}\bar{\boldsymbol{S}}' = -\boldsymbol{\Gamma}^{\mathrm{T}}. \tag{90}$$

Thus from (18), (85), (87), (88)

$$W_z = \bar{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{e} = \bar{\boldsymbol{f}}^{\mathrm{T}}\boldsymbol{f}. \tag{91}$$

This was proved by Suchy & Altman (1975). It must be independent of s, as was proved in § 3.

Near $s = 0$, the root of (81) that is infinite there is

$$q_1 \approx -a_3/\epsilon_{zz} \approx \Delta/s, \tag{92}$$

where (21), (29), (82) have been used. As an example, suppose that (92) is exactly true and that the medium is sufficiently slowly varying for the coupling terms in (87), (90) to be neglected. Then the first of the four equations (87) is

$$f_1' = -\mathrm{i}(\Delta/s)f_1, \tag{93}$$

with the solution
$$f_1 = P\exp(-\mathrm{i}\Delta\ln s) \tag{94}$$

(compare first term of (33) for solution (37)). Take Δ to be positive. Then the phase propagation of this wave is upwards when s is real and positive and downwards when it is real and negative. But it is explained later, § 10 that the direction of group propagation or ray is obliquely upwards for all real $s \neq 0$. Similarly for the adjoint equation (90) we may choose the solution

$$\bar{f}_1 = P^* \exp(+\mathrm{i}\Delta\ln s) = f_1^* \quad (s \text{ real and positive}). \tag{95}$$

Then, from (19), for s real and positive, W_z is $4Z_0\Pi_z$ where Π_z is the time average of the z component of the Poynting vector. By arguments similar to those in §§ 6 and 7 it now follows that

$$\Pi_z(-) - \Pi_z(+) = PP^*(\mathrm{e}^{2\pi\Delta}-1)/4Z_0. \tag{96}$$

Thus there is a disappearance of some energy at the resonance, and it has now been shown that this occurs for the characteristic wave that is singular at the resonance.

This argument fails, however, if $|\Delta|$ is small or zero, for then there is a coupling point near to or at $s = 0$. Since the coupled equations (87) fail at a coupling point we need an alternative to the transformation (86) with a new matrix $\boldsymbol{V}$ replacing $\boldsymbol{S}$, such that the resulting equations can be used in a domain containing both a resonance and coupling point.

A transformation leading to equations that can be used in a domain containing a coupling point was given by Heading (1961*a*) and adapted for the radio propagation application by Budden (1972). Instead of the matrix $\boldsymbol{S}$ in (86), it used a new matrix $\boldsymbol{U}$ analytic and non-singular at a coupling point, but unfortunately some of its elements have poles at a resonance. A suitable new transforming matrix $\boldsymbol{V}$ is constructed as follows.

We are interested in only two roots q_1, q_2 of the quartic (81) and their associated eigen columns $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ of $\boldsymbol{T}$. Let q_1 be the one that is singular at $s = 0$ when $\Delta \neq 0$. Consider a domain of the complex s-plane containing a resonance at $s = 0$, and two coupling points, at s_{c1}, s_{c2} which are real and satisfy

$$s_{c2} < s_{c1} \leqslant 0. \tag{97}$$

At both these points $q_1 = q_2$ and they will form the boundaries of the barrier. It was shown by Budden (1972) that $q_1 + q_2$ and $(q_1 - q_2)^2$ are analytic at a coupling point, and that $q_1 - q_2$ has a simple branch point there.

It is thus assumed that q_1, q_2 are not equal to q_3, q_4 anywhere in the domain. The waves associated with q_3, q_4 are independently propagated. The coupling terms involving them are assumed to be small and are neglected. A coupling point where $q_3 = q_4$ within the domain could be dealt with but is not of interest here.

The matrix $\boldsymbol{S}$, (86) used by Budden & Clemmow (1957) and *R.w.i.* (18.74) is bounded at a resonance but singular at a coupling point. In addition each of its columns $\boldsymbol{s}_i$ contains a normalizing factor, $(A_i F_i)^{-\frac{1}{2}}$ in their notation, that has a branch point like $(s - s_c)^{-\frac{1}{4}}$ at each coupling point. These factors are therefore omitted and a factor ϵ_{zz}^{-1} is also omitted. The resulting new matrix will be called $\mathscr{S}$. It could still be used to derive a form of (87) but it does not satisfy (89). A column of $\mathscr{S}$ is given in the appendix. Its elements contain integer powers of q up to q^3. Its first column is infinite like s^{-2} at $s = 0$. The columns of $\mathscr{S}$ will now be denoted by $\boldsymbol{s}_i$.

The third and fourth columns of $\mathscr{S}$ can be used as the third and fourth columns of the new matrix $\boldsymbol{V}$. The first two columns of $\boldsymbol{V}$ are to be suitable independent linear combinations of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. First consider $\boldsymbol{s}_1 - \boldsymbol{s}_2$. This has a factor $q_1 - q_2$ and so take

$$(\boldsymbol{s}_1 - \boldsymbol{s}_2)/(q_1 - q_2). \tag{98}$$

This is analytic at each coupling point and was used by Budden (1972) as the second column of $\boldsymbol{U}$. But its elements contain terms $q_1 + q_2$ and $\epsilon_{zz}(q_1^2 + q_1 q_2 + q_2^2)$ which are infinite like ϵ_{zz}^{-1} at the resonance. Hence for the second column of $\boldsymbol{V}$ we take

$$(\boldsymbol{s}_1 - \boldsymbol{s}_2)\,\epsilon_{zz}/(q_1 - q_2). \tag{99}$$

This is bounded and analytic at both the resonance and the coupling points.

Finally consider $-q_2\boldsymbol{s}_1 + q_1\boldsymbol{s}_2$. This also has a factor $q_1 - q_2$, and so take

$$(-q_2\boldsymbol{s}_1 + q_1\boldsymbol{s}_2)/(q_1 - q_2). \tag{100}$$

This too is analytic at a coupling point and its elements contain terms $q_1 q_2$ and $\epsilon_{zz} q_1 q_2(q_1 + q_2)$ which again are infinite like ϵ_{zz}^{-1} at the resonance. This is still true in the limit $\Delta = 0$. Hence for the first column of $\boldsymbol{V}$ we take

$$(-q_2\boldsymbol{s}_1 + q_1\boldsymbol{s}_2)\,\epsilon_{zz}/(q_1 - q_2). \tag{101}$$

Thus the required matrix $\boldsymbol{V}$ is given by

$$\boldsymbol{V} = \mathscr{S}\boldsymbol{W}, \quad \boldsymbol{W} = \begin{bmatrix} -q_2 p & p & 0 & 0 \\ q_1 p & -p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{102}$$

where p means $\epsilon_{zz}/(q_1 - q_2)$.

Define four new independent variables as a column $\boldsymbol{g}$ satisfying

$$\boldsymbol{e} = \boldsymbol{V}\boldsymbol{g} \tag{103}$$

(compare (86)). Substitute in (14). This gives

$$\boldsymbol{g}' = -\mathrm{i}\boldsymbol{A}\boldsymbol{g} + \boldsymbol{\Lambda}\boldsymbol{g}, \quad \boldsymbol{\Lambda} = -\boldsymbol{V}^{-1}\boldsymbol{V}', \tag{104}$$

where

$$\boldsymbol{T}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{A}, \quad \boldsymbol{A} = \boldsymbol{W}^{-1}\boldsymbol{Q}\boldsymbol{W} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -q_1 q_2 & q_1 + q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}. \tag{105}$$

Now $\boldsymbol{V}$ is bounded and non-singular throughout the domain. The new coupling matrix $\boldsymbol{\Lambda}$ contains $\boldsymbol{V}'$ which is small in a slowly varying medium. It is now assumed that the medium varies slowly enough, and that g_3, g_4 are small enough for the coupling terms $\boldsymbol{\Lambda}\boldsymbol{g}$ in (104) to be neglected. Then the first two of the four equations (104) are separate from the others, and give

$$\partial g_1/\partial s = -\mathrm{i}g_2, \tag{106}$$

$$\partial g_2/\partial s = \mathrm{i}q_1 q_2 g_1 - \mathrm{i}(q_1 + q_2)\,g_2, \tag{107}$$

whence

$$\partial^2 g_1/\partial s^2 + \mathrm{i}(q_1 + q_2)\,\partial g_1/\partial s - q_1 q_2 g_1 = 0. \tag{108}$$

These equations can now be used to study tunnelling through a barrier when there is a resonance nearby.

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10. A model for a barrier with a nearby resonance

A model is now to be devised that displays the main features of resonance tunnelling. It must apply for obliquely incident waves as well as for normal incidence. The coordinate axes can be chosen so that $S_2 = 0$, $S_1 = S$, which means that the wave normals are in the $x-z$ plane. The use of z rather than $s = kz$ will here be resumed.

The waves associated with the roots q_1, q_2 of (81) are to be studied, and those associated with q_3, q_4 are assumed to be independently propagated. A factor $(q-q_3)(q-q_4)$ can therefore be divided out from (81) leaving a quadratic whose roots are q_1, q_2. This is to have an infinite root at $z = 0$ and equal roots at two real negative values of z. A suitable quadratic is

$$zq^2 - wSq + z(S^2-b^2) - S^2 w \cot 2\psi - v = 0 \tag{109}$$

where b, v, w, ψ are positive constants with

$$v > \tfrac{1}{2}wb^2 \tan\psi, \quad \psi < \tfrac{1}{2}\pi. \tag{110}$$

For normal incidence, $S = 0$, $q = \mu$ this gives

$$\mu^2 = b^2 + v/z, \tag{111}$$

which is just the degenerate case studied in *R.w.i.* § 21.15. The inclusion of $S \neq 0$ now permits removal of the degeneracy. Typical curves of q versus $s = kz$ are shown in figure 1 for various $S \propto \Delta$.

Now S, q are the x, z coordinates of the refractive index μ regarded as a vector in the direction of the wave normal. A curve of q versus S, for a fixed z, is a cross section of the refractive index surface by a plane parallel to the $x-z$ plane. Figure 2 shows a family of such curves for various values of z. They are all conics with a common axis at an angle $\pi-\psi$ to the S axis. Their properties are in many respects similar to those for the Extraordinary radio wave in the ionosphere when the frequency exceeds the electron gyro frequency.

When $z \to -\infty$ the curve is a circle of radius b. It simulates an isotropic medium such as the free space below the ionosphere. As z increases the curves become ellipses which get smaller. When $z = -v/b^2$ they shrink to a point. This is just what happens to the Extraordinary wave refractive index surfaces at the cut off $X = 1-Y$ (Ratcliffe 1959). For $-v/b^2 < z < -\tfrac{1}{2}w\tan\psi$ there are no real curves. For $-\tfrac{1}{2}w\tan\psi < z < \tfrac{1}{2}w\cot\psi$ the curves are hyperbolae. They simulate the (more complicated) curves for the Z mode in the magnetoionic case when $1-Y^2 < X < 1$. For $z > \tfrac{1}{2}w\cot\psi$ the curves are ellipses. They now simulate the Z mode for $1 < X < 1+Y$. The Z mode cut off at $X = 1+Y$ is not simulated, but is outside the range of interest. When $z \to +\infty$ the curves approach a circle of radius b.

The family of refractive index surfaces can be used for ray tracing with Pöverlein's construction (Pöverlein 1948, 1949, 1950; *R.w.i.* § 13.21). A line is drawn perpendicular to the S axis at the value of S for the ray to be traced. Where it cuts a refractive index surface, the outward normal gives the direction of energy propagation, that is the ray. An example is shown as a chain line in figure 2. A typical intersection for positive z is at P. Here q is positive so the direction of phase propagation is upwards. The normal is obliquely upwards and this is the direction of energy flow. This point corresponds to branch 4 in figure 1*a*. Another intersection is at Q, where q is negative and here the normal is obliquely downwards so that both phase propagation and energy flow are downwards. This corresponds to branch 5 in figure 1*a*. When z is negative, a typical intersection is at R where q is negative so that the phase propagation is downwards. But the normal is is obliquely upwards and this is the direction of energy flow. This explains why the solution (94) represents an upgoing wave even when the effective $q \propto 1/s$ is negative. This point R corresponds to branch 3 in figure 1*a*. There is another intersection at S for the same negative z. Here the normal is obliquely downwards and so the wave is downgoing, branch 6 in figure 1*a*.

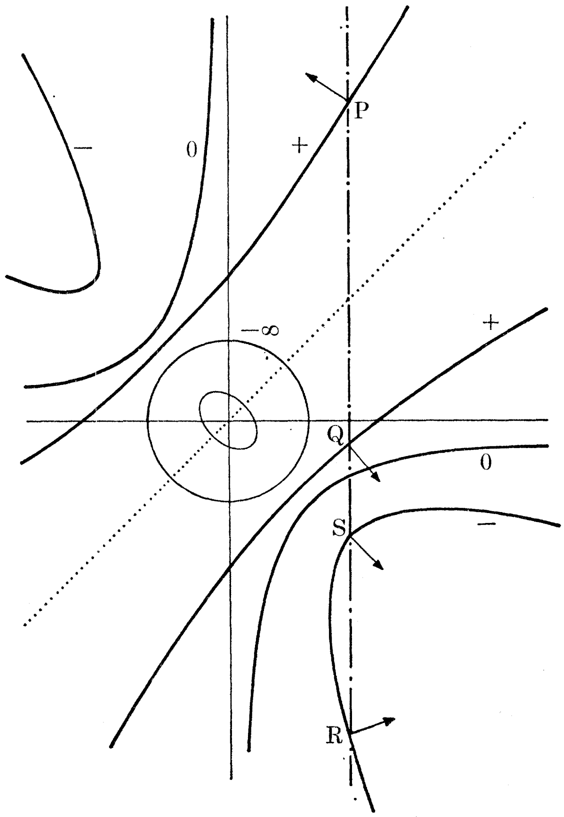

FIGURE 2. Family of refractive index surfaces as given by the model, (109), for various values of z. The symbols $+$, 0, $-$, $-\infty$, by the curves refer to the values of z. The chain line is used in Pöverlein's construction.

It is now convenient to study the special case $\psi = \frac{1}{4}\pi$. The inclusion of a general ψ complicates the algebra but does not reveal any new physical principles. Hence (109) becomes

$$zq^2 - wSq + z(S^2 - b^2) - v = 0, \tag{112}$$

and the first condition (110) is

$$2v > wb^2. \tag{113}$$

If this is violated, all the curves of the family go through a common point. Though less realistic, this case is of some interest and is mentioned again in § 12.

Also of interest is a variant of (112) in which the constant b is replaced by S to give

$$zq^2 - wSq - v = 0. \tag{114}$$

This equation has only one coupling point, where

$$z = -\tfrac{1}{4}w^2S^2/v. \tag{115}$$

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It is used in § 11 to study the effect of a single coupling point with a nearby resonance. If $S = 0$, (114) gives

$$\mu^2 = v/z \tag{116}$$

which is the degenerate case studied in *R.w.i.*, § 21.14.

11. Single coupling point and nearby resonance

The object of this section is to examine how the behaviour of the waves near a coupling point is modified when there is a resonance nearby. The properties of a well isolated coupling point were examined by Budden (1972) who showed that the solutions can be expressed in terms of Airy integrals. This is no longer true when a resonance is near.

It is therefore assumed that q_1, q_2 satisfy (114). The coupling point is given by (115). Substitution of (114) in (108) gives

$$z\frac{d^2 g_1}{dz^2} + ikwS\frac{dg_1}{dz} + kvg_1 = 0. \tag{117}$$

The general solution (Watson 1944, p. 97, eqn. (7)) is

$$g_1 = z^{\frac{1}{2}\nu}\mathscr{C}_\nu\{2k(vz)^{\frac{1}{2}}\}, \tag{118}$$

where $\mathscr{C}$ is any Bessel function and $\nu = 1 - ikSw.$ (119)

For $S = 0$, $\nu = 1$, (118) is the same as (69) and was used in *R.w.i.*, (21.64).

Now seek a solution (118) which includes a wave travelling in the direction of decreasing z (downwards) where z is large and positive, and which contains no upgoing wave when z is large and negative. The required solution uses the Hankel function $\mathscr{C} = H^{(1)}$, as in *R.w.i.*, § 21.14. When its asymptotic approximations (Watson 1944, p. 197) are used in (118) they give

$$z \text{ large and positive: } g_1 \sim z^p \exp\{2ik(vz)^{\frac{1}{2}}\} \tag{120}$$

$$z \text{ large and negative: } g_1 \sim z^p \exp\{-2kv^{\frac{1}{2}}|z|^{\frac{1}{2}}\} \tag{121}$$

where $p = \frac{1}{4} - \frac{1}{2}ikSw$.

Here (120) is the incident downgoing wave. There is no other term and therefore no reflected wave. On the other side of the plasma (121) is an evanescent wave and gets indefinitely small when $|z|$ is large. Thus all the incident energy apparently disappears.

The dependence of q on z is similar to the two right hand curves of figure 1*a* if S is positive, or figure 1*d* if S is negative. These figures can be used in a rather crude physical description of what happens. The incident wave (120) is represented by branch 5. In figure 1*a* it travels down to branch 6 where $z < 0$, gets reflected at B, becomes an upgoing wave represented by branch 3, and finally disappears at the resonance. In figure 1*d*, branch 5, it simply encounters the resonance and disappears before it can get reflected. To examine these effects, a more detailed study must be made of the solution (118).

Let $$\zeta = 2k(vz)^{\frac{1}{2}}. \tag{122}$$

Figure 3 shows the complex ζ/ν plane. If $|\zeta|$ and $|\nu|$ are both $\gg 1$, there are regions near $\zeta/\nu = \pm 1$ shown shaded in figure 3, where the Bessel functions $\mathscr{C}_\nu(\zeta)$ may be approximated by Airy integrals (Olver 1954). The coupling point (115) is where

$$\zeta = ikw|S|, \quad \zeta/\nu = ikw|S|/(1 - ikwS). \tag{123}$$

The locus of this for varying real S is two semi-circles in the upper half ζ/ν plane.

The real z axis maps into two perpendicular straight lines through the origin with directions depending on S. If S is large and negative they are as shown as a1 (z positive) and a2 (z negative) in figure 3. The coupling point is on a2 and near to where $\zeta/\nu = 1$, from (123). It is at Ca in the shaded region of figure 3, and a long way from the resonance because S is large. Thus it might be regarded with good approximation as well isolated. Then the treatment of Budden (1972) can be used with Airy integral solutions. There is no need to use the Bessel function. Similarly the resonance is well isolated, and can be treated separately as in §9 at (93), (94). Elsewhere, in regions remote from the resonance and the coupling point, the Airy integrals can be expressed as asymptotic approximations, but these are simply the W.K.B. solutions of the original equations (14).

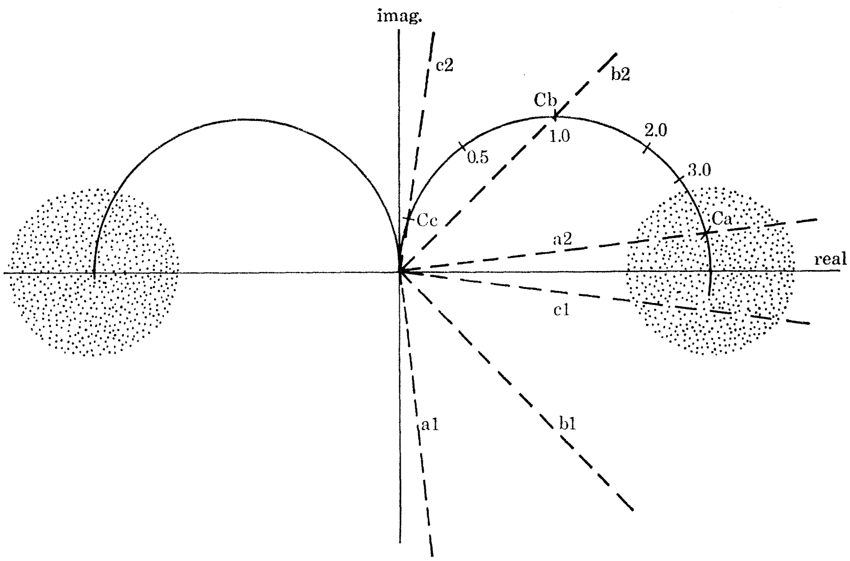

FIGURE 3. The complex ζ/ν plane. The resonance is at the origin and the coupling point must lie on one of the two semicircles shown as continuous curves. In regions like those shown shaded (not drawn to scale) any Bessel function $\mathscr{C}_\nu(\zeta)$ may be approximated by Airy integrals, provided that $|\zeta|$ and $|\nu|$ are large enough. The real z axis for positive z maps into the broken lines a1 or b1 or c1 according as S is large, intermediate or small respectively. Similarly the broken lines a2, b2 and c2 are the mapping of the real z axis for negative z, for the same three values of S. The numbers near marked points are the values of $|\zeta|$.

Suppose, next, that S is negative but not large. Then the real z axis maps into the lines b1 (z positive) and b2 (z negative) in figure 3. The coupling point is now at Cb and is not in the shaded region. It occurs where $|\zeta|$ and $|\nu|$ are both of order unity, which is too small to allow the Bessel function to be expressed in terms of its asymptotic forms alone. It cannot be treated as isolated because its properties are now modified by the proximity of the resonance. Within a domain containing both the resonance and the coupling point, the characteristic waves have lost their identity and the Bessel function cannot be dispensed with. Outside the domain, the W.K.B. solutions can nearly always be used, provided certain precautions are taken. Heading (private communication) has pointed out that there can sometimes be serious difficulties in linking the solutions in different domains by means of the W.K.B. solutions. To locate the boundaries of the domain it would be necessary to fix error bounds on the use of the W.K.B. solutions, but this is not needed in the present discussion.

Finally let S be negative and very small. Then the real z axis maps into the lines c1 (z positive) and c2 (z negative) in figure 3. Now c1 passes through a shaded region but here $|\zeta|$ and $|\nu|$ are close to unity and therefore too small for the Airy integral approximations to be applicable. The coupling point is at Cc very close to the resonance at the origin. Again it is essential to use the Bessel function when $|\zeta/\nu|$ does not greatly exceed unity. This case approaches the degenerate case $S = 0$ studied in *R.w.i.* and in §7.

If S is positive, the behaviour is very similar, except that the shaded region near $\zeta/\nu = -1$ is now used.

12. Barrier with nearby resonance

We now tackle the full problem where there is a barrier with a resonance at or just beyond one boundary. It is assumed that q_1, q_2 satisfy (112) with $S < b$. Their behaviour is shown in figure 1. Substitution in (108) gives

$$\frac{d^2 g_1}{dz^2} + \frac{ikSw}{z}\frac{dg_1}{dz} + k^2\left(\frac{v}{z} + b^2 - S^2\right) g_1 = 0. \tag{124}$$

Change the dependent variable to

$$h = g_1 z^p, \quad \text{where} \quad p = \tfrac{1}{2}ikSw, \tag{125}$$

and change the independent variable to

$$\zeta = 2ik\gamma z, \quad \text{where} \quad \gamma = (b^2 - S^2)^{\frac{1}{2}} \tag{126}$$

and γ is positive. This gives

$$\frac{d^2 h}{d\zeta^2} + \left\{-\tfrac{1}{4} + \frac{\kappa}{\zeta} + \frac{\frac{1}{4} - m^2}{\zeta^2}\right\} h = 0, \tag{127}$$

where

$$\kappa = -\tfrac{1}{2}ikv/\gamma, \quad m = \pm\tfrac{1}{2}(1 - ikSw). \tag{128}$$

This is the form given by Whittaker & Watson (1935, p. 337) for the confluent hypergeometric function.

(i) *Waves incident from direction of negative z*

One solution of (127) is

$$h = W_{\kappa,m}(\zeta) \tag{129}$$

which is the same as used in *R.w.i.*, §21.15 except that there S was zero and m was $\pm\frac{1}{2}$. When z is large and positive (126) shows that $\arg\zeta = \frac{1}{2}\pi$, and the asymptotic form for (129) gives (Whittaker & Watson 1935, p. 343) with (125)

$$g_1 \sim e^{-\frac{1}{2}\zeta}\zeta^{\kappa} z^{-p} = \exp\{-ik\gamma z - \tfrac{1}{2}ik(Sw + v/\gamma)\ln z - \tfrac{1}{2}ik(v/\gamma)\ln(2k\gamma) + \tfrac{1}{4}\pi kv/\gamma\}. \tag{130}$$

This represents an outgoing wave. There is no other term and therefore no incoming wave.

When z is large and negative, it follows from (126) and (13) that $\arg\zeta = 3\pi/2$. The asymptotic approximation, given by Heading (1961 *b*), is

$$W_{\kappa,m}(\zeta) \sim e^{-\frac{1}{2}\zeta}\zeta^{\kappa} + \frac{2\pi i e^{2\pi i\kappa} e^{\frac{1}{2}\zeta}\zeta^{-\kappa}}{\Gamma(\frac{1}{2} - m - \kappa)\,\Gamma(\frac{1}{2} + m - \kappa)}. \tag{131}$$

The first term is similar to (130) but now, from (13), $\ln z$ is $\ln|z| + i\pi$. With (125) this gives

$$g_1 \sim \exp\{-ik\gamma z - \tfrac{1}{2}ik(Sw + v/\gamma)\ln|z| - \tfrac{1}{2}ik(v/\gamma)\ln(2k\gamma) + \tfrac{1}{4}\pi kv/\gamma\}\exp\{\tfrac{1}{2}\pi k(Sw + v/\gamma)\}. \tag{132}$$

This is an incoming wave, the incident wave. Define the transmission coefficient $|T|$ as the ratio of the moduli of (130) for z large and positive, to (132) for the same z but negative. Then

$$|T| = \exp\{-\tfrac{1}{2}\pi k(Sw + v/\gamma)\}. \tag{133}$$

The second term of (131) contributes

$$g_1 \sim \frac{2\pi i}{\Gamma(\frac{1}{2}-m-\kappa)\,\Gamma(\frac{1}{2}+m-\kappa)} \exp\{ik\gamma z - \tfrac{1}{2}ik(Sw-v/\gamma)\ln|z| + \tfrac{1}{2}ik(v/\gamma)\ln(2k\gamma) + \tfrac{1}{4}\pi kv/\gamma + \tfrac{1}{2}\pi kSw\}. \quad (134)$$

This is an outgoing wave, the reflected wave. Define the reflexion coefficient $|R|$ as the ratio of the moduli of (134) to (132). Then with (128)

$$|R| = \frac{2\pi \exp(-\frac{1}{2}\pi kv/\gamma)}{|\Gamma\{\frac{1}{2}ik(Sw+v/\gamma)\}\,\Gamma\{1-\frac{1}{2}ik(Sw-v/\gamma)\}|}$$
$$= \left[\frac{v/\gamma+Sw}{v/\gamma-Sw}\{1-2e^{-\pi kv/\gamma}\cosh(\pi kSw)+e^{-2\pi kv/\gamma}\}\right]^{\frac{1}{2}}, \quad (135)$$

where the formula $|\Gamma(ix)| = \{(x\sinh\pi x)/\pi\}^{-\frac{1}{2}}$ has been used.

In the special case $S = 0$, (133), (135) give

$$|T| = e^{-\frac{1}{2}\pi kv/\gamma}, \quad |R| = 1-e^{-\pi kv/\gamma} \quad (136)$$

which agree with *R.w.i.*, (21.75) and (21.76). Here $|R|$ and $|T|$ were defined in terms of the amplitudes contributed by the waves to g_1. They are not simply related to the energy fluxes, as wrongly assumed in *R.w.i.*, but the conclusion that some energy disappears is still true.

The equation

$$Sw = -v/\gamma \quad (137)$$

cannot be satisfied by any real S if (113) is true. Let us temporarily discard the restriction (113). Then, when (137) is satisfied (135) and (133) show that $|R| = 0$ and $|T| = 1$. The wave just goes straight through without reflexion. But (112) shows that (137) is the condition that the width of the barrier is zero, and one q has the constant value γ for all z. The two coupling points have moved to a coalescence, of the type which Budden & Smith (1974) call C2. In this example, the wave now behaves as though the medium is homogeneous. It is the mode that is unaffected by the resonance and so there can now be no disappearance of energy.

When

$$Sw = +v/\gamma \quad (138)$$

which, again, can only be true for real S if (113) is discarded, then (135) and (133) show that

$$|R|^2 = \{1-\exp(-2\pi kv/\gamma)\}\,2\pi kv/\gamma, \quad |T| = \exp(-2\pi kv/\gamma). \quad (139)$$

Again there is a coalescence type C2 of the coupling points. One q has the constant value $-\gamma$ for all z, but it is now the other value of q which applies to the wave, and has the resonance. Budden & Smith (1974) showed that for an isolated coalescence C2 there should be no reflected wave. Here there is some reflexion (139) because the coalescence is not sufficiently isolated. Its behaviour is modified by the proximity of the resonance.

(ii) *Waves incident from direction of positive z*

Instead of the first equations (126), (128) take

$$\xi = -2ik\gamma z, \quad \kappa = +\tfrac{1}{2}ikv/\gamma. \quad (140)$$

If these, with (125), are substituted in (124) there results an equation the same as (127) except that ζ is replaced by ξ. One solution is

$$h = W_{\kappa,m}(\xi). \quad (141)$$

When z is large and positive, $\arg\xi = -\frac{1}{2}\pi$ and the asymptotic form, (Whittaker & Watson 1935, p. 343) gives

$$h \sim e^{-\frac{1}{2}\xi}\xi^{\kappa} \tag{142}$$

$$g_1 \sim \exp\{ik\gamma z - \tfrac{1}{2}ik(Sw - v/\gamma)\ln z + \tfrac{1}{2}ik(v/\gamma)\ln(2k\gamma) + \tfrac{1}{4}\pi kv/\gamma\}. \tag{143}$$

This is an incoming wave, the incident wave. There is no other term and therefore no reflected wave. When z is large and negative, $\arg\xi = +\frac{1}{2}\pi$ and the asymptotic form is still (142), whence

$$g_1 \sim \exp\{ik\gamma z - \tfrac{1}{2}ik(Sw - v/\gamma)\ln|z| + \tfrac{1}{2}ik(v/\gamma)\ln(2k\gamma) + \tfrac{1}{2}\pi kSw - \tfrac{1}{4}\pi kv/\gamma\}. \tag{144}$$

This is an outgoing wave, the transmitted wave. The ratio of the moduli of (144) to (143) gives the modulus of the transmission coefficient

$$|T| = \exp\{\tfrac{1}{2}\pi k(Sw - v/\gamma)\}. \tag{145}$$

If $S = 0$, this agrees with *R.w.i.*, (21.82).

If (138) is true, which is only possible for real S when (113) is discarded, (145) gives $|T| = 1$. One q has the constant value $-\gamma$ for all z, and $|T|$ refers to this wave. It is unaffected by the other q which is the one that has the resonance. This is another example of coalescence C2 of the coupling points.

The foregoing treatment is similar to that in *R.w.i.*, § 21.15, but the degeneracy has been removed. The reflexion and transmission coefficients (133), (135) and (145) apply both for obliquely incident and normally incident waves.

13. What happens to the lost energy?

The question of what happens to the wave energy that disappears near a resonance is an intriguing physical problem. At least three different view points may be taken.

First it may be argued that the assumption of a loss free medium is unrealistic. There is no medium, except for a vacuum, from which all forms of wave attenuation are absent. Therefore the refractive index μ always has a non-zero imaginary part when $s = kz$ is real. In a plasma this could be caused by collision damping, Landau damping or other forms of damping. Im (μ) may be negligibly small for most real values of z. But the pole at the resonance is not exactly on the real z axis. Near it both real and imaginary parts of μ are very large, so that the energy absorption rate is large. The lost energy is simply converted to heat or other forms of energy in the region of the real z axis close to the pole. In this paper nonlinear effects have been ignored. The solutions for E_x, E_y, proportional to (33) are bounded near the resonance, so that for these components, the assumption may be justified for small wave amplitudes. But (25) shows that E_z is not bounded at the resonance. Thus non-linear effects must occur. One result would be the generation of harmonics of the wave frequency. This provides another way by which wave energy could be lost.

Second, the illustrations in this paper apply for a cold plasma whose dielectric constant is given by (3). If the plasma is warm its dielectric constant is not given exactly by (3). The result is that when ϵ_{zz} approaches zero, one refractive index gets very large, but it does not become infinite when $\epsilon_{zz} = 0$. Instead, the characteristics of the wave go over continuously to those of a plasma wave. Curves showing how this happens have been given by Ginzburg (1970, Figs 12.2–12.4, 12.9–12.13) and the subject has been studied by Stix (1965). On this view the resonance does not exist. The waves have simply gone over to a mode of propagation governed by a modification of the basic equation, and they carry the 'lost' energy with them. This is not a mode conversion. The transition is continuous and it is only the name of the wave type that changes. There can be mode conversion near the transition, but this is a separate phenomenon associated with a coupling point.

If either of these views is taken, the lost energy can be accounted for and there is nothing more to be said. But it is still of interest to pursue the enquiry for a fictitious medium which is a cold plasma and really has no damping mechanism at all. The following suggestion is offered. In any wave system some energy is stored in the medium. The time average of the stored energy per unit volume is constant in the steady state conditions assumed here, and is proportional to the dielectric constant and to the square of the electric field. Near a resonance at $z = 0$ the dielectric constant (3) is bounded, but (25) shows that E_z tends to infinity like $1/z$. Consequently if the stored energy is integrated through a volume near the resonance, the integral gets indefinitely large as the edge of the volume moves up to the resonance. The total energy stored in the part of the medium containing the resonance is infinite. But this state of affairs cannot be attained in a finite time. After the wave is first switched on, a state of physical equilibrium is never reached. The medium has an infinite capacity for storing energy. The energy that is apparently lost is simply going into storage in the medium.

14. Conclusions

This paper has attempted to study the physical processes that occur for a wave system in a medium containing a resonance: that is, an infinity of one of the refractive indices. The waves are assumed to be of small amplitude so that nonlinear effects can be ignored. In a slowly varying medium the characteristic waves are independently propagated at most points, but this is not true at coupling points. The behaviour of waves near a coupling point, or near two adjacent coupling points is already well understood. It is found to be modified if there is a resonance nearby.

Propagation in a stratified system is conveniently described in terms of the variable q, (79) and (83). Near an ideal isolated coupling point $(q_1-q_2)^2$ is a linear function of z and the wave fields can be expressed in terms of an Airy integral, Ai (Kz), where K is a constant. If $(q_1-q_2)^2$ is not exactly linear it can be used to define a new variable ζ which is a monotonic function of z, and the fields can be expressed in terms of Ai (ζ). This is the principle of uniform approximation. The Airy integral is used as a comparison function. If two coupling points, for the same two characteristic waves, are close together, $(q_1-q_2)^2$ is close to a quadratic function of z, and the required comparison function is a Weber function (Rydbeck 1943; Budden & Smith 1974). This paper has considered a domain containing one or two such coupling points and a resonance at which one of q_1, q_2 is infinite. If this domain is well enough isolated from other coupling points and resonances involving q_1, q_2, then q_1 and q_2 are the roots of a quadratic equation (109) or (112). In these examples the coefficients in the quadratic are linear functions of z. In practical cases the z dependence would not be exactly linear and z would have to be replaced by ζ, a monotonic function of z. For a single coupling point and a resonance, the fields are expressed in terms of a Bessel function of complex order which would supply the comparison function. For two coupling points forming a barrier, with a resonance nearby, the comparison function is a Whittaker function.

It is not likely that the full uniform approximations in these cases will ever need to be worked out, since they would be very complicated. Complete solutions can be computed much more simply by using the basic equations, (14). But it is useful to know what the comparison functions are, because it is their properties, particularly their asymptotic properties, that are linked with the physics of the wave propagation.

Appendix

This paper has used eigen columns $\boldsymbol{s}_i$ of the matrix $\boldsymbol{T}$ (6) with eigen values q_i. The form of $\boldsymbol{s}_i$ used in § 9 onwards is given below. It contains elements of the dielectric constant tensor $\boldsymbol{\epsilon}$, (3), and also its principal axis components ϵ_1, ϵ_2, ϵ_3, Let

$$K_1 = lm\{\tfrac{1}{2}\epsilon_3(\epsilon_1+\epsilon_2)-\epsilon_1\epsilon_2\}+\mathrm{i}n\epsilon_3 D,$$

$$K_2 = \tfrac{1}{2}\epsilon_3(\epsilon_1+\epsilon_2)\,(m^2-1)-m^2\epsilon_1\epsilon_2+S_2^2\epsilon_{zz},$$

$$K_3 = mn\{\tfrac{1}{2}\epsilon_3(\epsilon_1+\epsilon_2)-\epsilon_1\epsilon_2\}-\mathrm{i}l\epsilon_3 D-S_1S_2\epsilon_{zx}-S_2^2\epsilon_{zy}.$$

In the following expression q may take any of the four values q_i. The four elements of $\boldsymbol{s}_i$ are

$$q^2S_1S_2+q(S_1\epsilon_{zy}+\epsilon_2\epsilon_{xz})-K_1-\epsilon_{zz}S_1S_2+(S_1^2+S_2^2)\,(\epsilon_{xy}+S_1S_2),$$

$$q^2(\epsilon_{zz}-S_2^2)-2qS_1\,ln\,G+K_2+(S_1^2+S_2^2)\,(\epsilon_{xx}-S_2^2),$$

$$q^3\epsilon_{zz}+q^2(S_2\epsilon_{zy}-2S_1\,ln\,G)+q(K_2+S_1^2\epsilon_{xx}+S_1S_2\epsilon_{xy})-S_2K_3,$$

$$q^2S_2\epsilon_{xz}-q\{K_1+S_1S_2G(l^2-n^2)-S_2^2\epsilon_{xy}\}+S_1K_3.$$

It can be checked that, in the special case $S_2 = 0$ and for a cold plasma with electrons only, these expressions are proportional to those used by Budden & Clemmow (1957) and in *R.w.i.* (18.61).

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